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Abstract. More than a decade ago, Moller and Tofts published their seminal work on relating processes that are annotated with lower time bounds, with respect to speed. Their paper has left open many questions concerning the semantic theory for their suggested bisimulation–based faster–than preorder, the MT–preorder, which have not been addressed since. The encountered difficulties concern a general compositionality result, a complete axiom system for finite processes, and a convincing intuitive justification of the MT–preorder.
This paper solves these difficulties by developing and employing novel tools for reasoning in discrete–time process algebra, in particular a general commutation lemma relating the sequencing of action and clock transitions. Most importantly, it is proved that the MT–preorder is fully–abstract with respect to a natural amortized preorder that uses a simple bookkeeping mechanism for deciding whether one process is faster than another. Together these results reveal the intuitive roots of the MT–preorder as a faster–than relation, while testifying to its semantic elegance. This lifts some of the barriers that have so far hampered progress in semantic theories for comparing the speed of processes.

Keywords. Asynchronous systems, timed process algebra, lower time bounds, faster–than relation, Moller–Tofts preorder, bisimulation.

1 Introduction

Over the past two decades, the field of process algebra [7] has proved successful for modeling and reasoning about the communication behavior of concurrent processes. Early process algebras, such as Milner's CCS [18] and Hoare's CSP [15], have been augmented to capture other important system aspects as well, including timing behavior [6]. Many variants of *timed process algebra* that employ either discrete or continuous notions of time have been proposed, whose semantic theories are usually based on the well–studied concepts of bisimulation [19], failures [22], or testing [14].

While several approaches for comparing the efficiency of processes have been introduced in the literature [4, 21], theories for comparing timed processes with

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respect to *speed* are seeded very sparsely. The most seminal paper in the latter category was published over a decade ago [20]. In this paper, the authors Moller and Tofts argue that a *faster–than relation* on processes can only exist for those process–algebraic settings where the passage of time cannot preempt behavior, and especially not for settings involving timeout operators. For a timeout–less fragment of TCCS [19], Moller and Tofts then introduced a compositional faster–than preorder based on strong bisimulation [18], and discussed some of its underlying algebraic laws. Despite the paper's originality, the work is lacking regarding three important aspects. First, the advocated preorder is not intuitively justified but appears to be an ad–hoc remedy for a compositionality problem. Second, the framework possesses technical weaknesses. For example, Moller and Tofts only managed to prove compositionality of their preorder for the class of regular processes, and their proposed laws for characterizing their preorder are incomplete. Third, no semantic theory that abstracts from internal computation, in the sense of observation equivalence [18], is presented in [20].

The aim of this paper is to put the faster–than preorder of Moller and Tofts, or *MT–preorder* for short, on solid semantic grounds and to highlight its intuitive roots, thereby testifying to the elegance of Moller and Tofts' approach. Technically, we add to Milner's CCS a discrete–time clock prefixing operator "$\sigma.$", interpreted as *lower time bound*. Intuitively, process P in $\sigma.P$ is only activated after the ticking of the abstract clock σ, i.e., after one time unit. The nesting of σ–prefixes then allows the specification of arbitrary delays (written as prefix (n) with $n \in \mathbb{N}$ in [20]), which results in a process algebra equivalent to the fragment of TCCS studied by Moller and Tofts. We refer to this algebra as *Timed Asynchronous Communicating Systems with lower time bounds*, or $\mathrm{TACS}^{\mathrm{LT}}$. As our first main result we prove that the MT–preorder is compositional and *fully–abstract* with respect to a natural *amortized* preorder that uses a simple bookkeeping mechanism for deciding whether one process is faster than another. The intuition behind this amortized preorder is that the faster process must execute each action no later than the slower process does, while both processes must be functionally equivalent in the usual sense of strong bisimulation. To obtain this result we also establish some powerful semantic tools for reasoning within discrete–time process algebra, in particular a general *commutation lemma* relating the sequencing of action and clock transitions. As our second main result we provide a sound and complete *axiomatization* of the MT–preorder for the class of finite processes. This includes the provision of a simple *expansion law*, which Moller and Tofts had claimed could not exist. The twist is that this expansion law is only valid for finite processes, but interestingly not for arbitrary recursive processes. As our third and final main result we introduce a notion of a *weak MT–preorder* — a task that turns out to be far more challenging than in other bisimulation–based process–algebraic settings.

Our results shed light on the nature of the MT–preorder and overcome the technical difficulties experienced by Moller and Tofts, thereby completing, generalizing, and strengthening their results and providing groundwork for advancing semantic theories that compare processes with respect to speed. This paper also complements our previous work on bisimulation–based faster–than relations for timed process algebra with *upper time bounds* [17]. Indeed, several ideas and technical concepts can be carried over from the upper–time–bounds setting of [17] to the lower–time–bounds setting presented here.

2 Timed Asynchronous Communicating Systems

Our process algebra $\mathrm{TACS}^{\mathrm{LT}}$ conservatively extends Milner's CCS [18] by permitting the specification of *lower time bounds* for the execution of actions and processes. These will then be used to compare processes with respect to speed. Syntactically, $\mathrm{TACS}^{\mathrm{LT}}$ includes a *clock prefixing operator* "$\sigma.$", taken from Hennessy and Regan's TPL [14]. Semantically, it adopts a concept of global, discrete time in which processes are *lazy* and can always let time pass. For example, $\sigma.P$ must wait for *at least* one time unit before it can start executing process P.

Syntax. The syntax of $\mathrm{TACS}^{\mathrm{LT}}$ is identical to the one in [17], where we considered a faster–than preorder that relates processes on the basis of upper rather than lower time bounds. Formally, let Λ be a countably infinite set of actions not including the distinguished unobservable, *internal* action τ. With every $a \in \Lambda$ we associate a *complementary action* $\overline{a}$. We define $\overline{\Lambda} =_{\mathrm{df}} \{\overline{a} \,|\, a \in \Lambda\}$ and take $\mathcal{A}$ to denote the set $\Lambda \cup \overline{\Lambda} \cup \{\tau\}$. Complementation is lifted to $\Lambda \cup \overline{\Lambda}$ by defining $\overline{\overline{a}} =_{\mathrm{df}} a$. As in CCS [18], an action a communicates with its complement $\overline{a}$ to produce the internal action τ. We let $a, b, \ldots$ range over $\Lambda \cup \overline{\Lambda}$, $\alpha, \beta, \ldots$ over $\mathcal{A}$, and represent clock ticks by σ. The syntax of $\mathrm{TACS}^{\mathrm{LT}}$ is defined as follows:

$$P \;\;::=\;\; \mathbf{0} \;\mid\; x \;\mid\; \alpha.P \;\mid\; \sigma.P \;\mid\; P + P \;\mid\; P|P \;\mid\; P \setminus L \;\mid\; P[f] \;\mid\; \mu x.P$$

where x is a *variable* taken from a countably infinite set $\mathcal{V}$ of variables, $L \subseteq \mathcal{A} \setminus \{\tau\}$ is a *restriction set*, and $f : \mathcal{A} \to \mathcal{A}$ is a *finite relabeling*. A finite relabeling satisfies the properties $f(\tau) = \tau$, $f(\overline{a}) = \overline{f(a)}$, and $|\{\alpha \,|\, f(\alpha) \neq \alpha\}| < \infty$. The set of all terms is abbreviated by $\widehat{\mathcal{P}}$, and we define $\overline{L} =_{\mathrm{df}} \{\overline{a} \,|\, a \in L\}$. Moreover, we use the standard definition for *open* and *closed* terms. A variable is called *guarded* in a term if each occurrence of the variable is within the scope of an action or clock prefix. Moreover, we require for terms of the form $\mu x.P$ that x is guarded in P. We refer to closed and guarded terms as *processes*, with the set of all processes written as $\mathcal{P}$, and write $\equiv$ for syntactic equality.

Semantics. The *operational semantics* of a $\mathrm{TACS}^{\mathrm{LT}}$ term $P \in \widehat{\mathcal{P}}$ is given by a labeled transition system $\langle \widehat{\mathcal{P}}, \mathcal{A} \cup \{\sigma\}, \longrightarrow, P \rangle$, where $\widehat{\mathcal{P}}$ is the set of states, $\mathcal{A} \cup \{\sigma\}$ the alphabet, $\longrightarrow \subseteq \widehat{\mathcal{P}} \times (\mathcal{A} \cup \{\sigma\}) \times \widehat{\mathcal{P}}$ the transition relation, and P the start state. Transitions labeled with an action α are called *action transitions* that, like in CCS, are local handshake communications in which two processes may synchronize to take a joint state change together. Transitions labeled with the clock symbol σ are called *clock transitions* representing a recurrent global synchronization that encodes the progress of time.

The operational semantics for action and clock transitions can be defined via the *structural operational rules* shown in Tables 1 and 2, respectively. As usual, we write $P \xrightarrow{\gamma} P'$ instead of $\langle P, \gamma, P' \rangle \in \longrightarrow$, for $\gamma \in \mathcal{A} \cup \{\sigma\}$, and say that P *may engage in* γ *and thereafter behave like* P'. Sometimes it is also convenient to write (i) $P \xrightarrow{\gamma}$ for $\exists P'. P \xrightarrow{\gamma} P'$, (ii) $\xrightarrow{\sigma}^k$ for $k \in \mathbb{N}$ consecutive clock transitions, with $\mathbb{N}$ including 0, and (iii) $P \xrightarrow{w} P'$, where either $w = \epsilon$ and $P \equiv P'$, or $w = \gamma w'$ for some $\gamma \in \mathcal{A} \cup \{\sigma\}$ and $w' \in (\mathcal{A} \cup \{\sigma\})^*$, and $\exists \hat{P}. P \xrightarrow{\gamma} \hat{P} \xrightarrow{w'} P'$.

Table 1. Operational semantics for TACS[LT] (action transitions)

Act	$\dfrac{-}{\alpha.P \xrightarrow{\alpha} P}$	Rel	$\dfrac{P \xrightarrow{\alpha} P'}{P[f] \xrightarrow{f(\alpha)} P'[f]}$	Rec	$\dfrac{P \xrightarrow{\alpha} P'}{\mu x.P \xrightarrow{\alpha} P'[\mu x.P/x]}$
Sum1	$\dfrac{P \xrightarrow{\alpha} P'}{P + Q \xrightarrow{\alpha} P'}$	Sum2	$\dfrac{Q \xrightarrow{\alpha} Q'}{P + Q \xrightarrow{\alpha} Q'}$	Res	$\dfrac{P \xrightarrow{\alpha} P'}{P \setminus L \xrightarrow{\alpha} P' \setminus L}\ \alpha \notin L \cup \overline{L}$
Com1	$\dfrac{P \xrightarrow{\alpha} P'}{P \vert Q \xrightarrow{\alpha} P' \vert Q}$	Com2	$\dfrac{Q \xrightarrow{\alpha} Q'}{P \vert Q \xrightarrow{\alpha} P \vert Q'}$	Com3	$\dfrac{P \xrightarrow{a} P' \quad Q \xrightarrow{\overline{a}} Q'}{P \vert Q \xrightarrow{\tau} P' \vert Q'}$

The *action–prefix* term $\alpha.P$ may engage in action α and then behave like P. It may also *idle*, i.e., engage in a clock transition to itself, as process $\mathbf{0}$ does. The *clock–prefix* term $\sigma.P$ can engage in a clock transition to P and ensures that there is a delay of *at least* one time unit before P is activated. The *summation operator* $+$ denotes nondeterministic choice: $P + Q$ may behave like P or Q; according to the deterministic nature of time, a clock transition cannot resolve choices. The *restriction operator* $\setminus L$ prohibits the execution of actions in $L \cup \overline{L}$ and, thus, permits the scoping of actions. $P[f]$ behaves exactly as P with actions renamed by the *relabeling* f. The term $P|Q$ stands for the *parallel composition* of P and Q according to an interleaving semantics with synchronized communication on complementary actions, resulting in the internal action τ. Again, time has to proceed equally on both sides of the operator, i.e., deterministically. Finally, $\mu x. P$ denotes *recursion*, it behaves as a distinguished solution to the equation $x = P$. The rules for action transitions are the same as for CCS, with the exception of the new clock–prefix operator and the rule for recursion; however, the latter is equivalent to the standard CCS rule over guarded terms [5].

The operational semantics for TACS[LT] possesses several important properties [14]. Firstly, any process can perform a clock transition due to our adoption of a lazy nil–process $\mathbf{0}$ and a lazy prefix operator. Secondly, the semantics is *time–deterministic*, i.e., progress of time does not resolve choices. Formally, $P \xrightarrow{\sigma} P'$ and $P \xrightarrow{\sigma} P''$ implies $P' \equiv P''$, for all $P, P', P'' \in \widehat{\mathcal{P}}$, which can easily be proved via induction on the structure of P.

Table 2. Operational semantics for TACS[LT] (clock transitions)

$$\text{tNil}\ \frac{-}{\mathbf{0} \xrightarrow{\sigma} \mathbf{0}} \qquad \text{tRec}\ \frac{P \xrightarrow{\sigma} P'}{\mu x.P \xrightarrow{\sigma} P'[\mu x.P/x]} \qquad \text{tRes}\ \frac{P \xrightarrow{\sigma} P'}{P \setminus L \xrightarrow{\sigma} P' \setminus L}$$

$$\text{tAct}\ \frac{-}{\alpha.P \xrightarrow{\sigma} \alpha.P} \qquad \text{tSum}\ \frac{P \xrightarrow{\sigma} P' \quad Q \xrightarrow{\sigma} Q'}{P + Q \xrightarrow{\sigma} P' + Q'} \qquad \text{tRel}\ \frac{P \xrightarrow{\sigma} P'}{P[f] \xrightarrow{\sigma} P'[f]}$$

$$\text{tPre}\ \frac{-}{\sigma.P \xrightarrow{\sigma} P} \qquad \text{tCom}\ \frac{P \xrightarrow{\sigma} P' \quad Q \xrightarrow{\sigma} Q'}{P|Q \xrightarrow{\sigma} P'|Q'}$$

3 The Moller–Tofts Preorder

This section first recalls the faster–than preorder introduced by Moller and Tofts in [20], to which we refer as *Moller–Tofts preorder*, or MT–preorder for short. As the section's main contribution, we prove the compositionality of this preorder for arbitrary processes, which has only been conjectured by Moller and Tofts. Indeed, the compositionality proof offered in [20] is restricted to processes that do not have any parallel operators inside the scope of a recursion. The key for proving compositionality in the general setting is a nontrivial *commutation lemma* that considers what happens when adjacent action and clock transitions are transposed. This lemma also plays an important role when obtaining the full–abstraction result presented in the next section.

Definition 1 (MT–preorder [20]). A relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is an *MT–relation* if, for all $\langle P, Q \rangle \in \mathcal{R}$ and $\alpha \in \mathcal{A}$:

1. $P \xrightarrow{\alpha} P'$ implies $\exists Q', k, P''.\, Q \xrightarrow{\sigma}{}^{k} \xrightarrow{\alpha} Q'$, $P' \xrightarrow{\sigma}{}^{k} P''$, and $\langle P'', Q' \rangle \in \mathcal{R}$.
2. $Q \xrightarrow{\alpha} Q'$ implies $\exists P'.\, P \xrightarrow{\alpha} P'$ and $\langle P', Q' \rangle \in \mathcal{R}$.
3. $P \xrightarrow{\sigma} P'$ implies $\exists Q'.\, Q \xrightarrow{\sigma} Q'$ and $\langle P', Q' \rangle \in \mathcal{R}$.
4. $Q \xrightarrow{\sigma} Q'$ implies $\exists P'.\, P \xrightarrow{\sigma} P'$ and $\langle P', Q' \rangle \in \mathcal{R}$.

We write $P \sqsupseteq\!\!\!\!\sim_{\mathrm{mt}} Q$ if $\langle P, Q \rangle \in \mathcal{R}$ for some MT–relation $\mathcal{R}$, and call $\sqsupseteq\!\!\!\!\sim_{\mathrm{mt}}$ the *MT–preorder*.

Technically, all conditions of this definition, with the exception of the first one, are identical to the ones of *temporal strong bisimulation* (cf., e.g., [8]). Intuitively, the weaker first condition states that, if the faster process P can perform an action, then the slower process Q must not match this action right away, but can perform an arbitrary number k of time steps before doing so. However, delaying k time steps may make the resulting process Q' faster than P'. To account for this, Moller and Tofts suggest that P' performs k time steps of its own, resulting in process P'' that should then be faster than Q'. To see the necessity for this, consider the processes $a.\mathbf{0}|\sigma.b.\mathbf{0}$ and $\sigma.a.\mathbf{0}|\sigma.b.\mathbf{0}$, for which a sensible faster–than preorder should clearly identify the former process as the faster one. Here, the a–transition of the former process to $\mathbf{0}|\sigma.b.\mathbf{0}$ can only be matched by the latter process after a delay of one time unit, leading to $\mathbf{0}|b.\mathbf{0}$. However, $\mathbf{0}|\sigma.b.\mathbf{0}$ is not faster than $\mathbf{0}|b.\mathbf{0}$, but only if it has delayed a time unit as well. *Forcing* the faster process to match the delay of the slower one *immediately* seems arbitrary and restrictive. Nevertheless, we will show in the next section that this is not the case and that there is a very natural explanation for this.

It is easy to see that $\sqsupseteq\!\!\!\sim_{\mathrm{mt}}$ is indeed a preorder, i.e., it is reflexive and transitive, and that it is the largest MT–relation. Moreover, if one studies CCS process terms only, i.e., TACS$^{\mathrm{LT}}$ processes not containing any clock prefix operator, then two processes are related in the MT–preorder if and only if they are strongly bisimular. This is because here all clock transitions are idling transitions, i.e., σ–loops; vice versa, every process can idle due to the laziness property. Hence, CCS is a sub–calculus of TACS$^{\mathrm{LT}}$.

Theorem 2 (Precongruence). *The MT–preorder* $\sqsupseteq\!\!\!\sim_{mt}$ *is a precongruence for all* TACS$^{\mathrm{LT}}$ *operators.*

The only difficult and non–standard part of the proof concerns compositionality regarding parallel composition and is based on the following *commutation lemma*.

Lemma 3 (Commutation). *Let* P, $P' \in \mathcal{P}$ *and* $w \in (\mathcal{A} \cup \{\sigma\})^*$.

1. Simple commutation lemma: *If* $P \xrightarrow{w}\xrightarrow{\sigma} P'$, *then* $\exists P''.\, P \xrightarrow{\sigma}\xrightarrow{w} P''$ *and* $P' \sqsupseteq\!\!\!\sim_{mt} P''$.
2. Commutation lemma: *If* $P \xrightarrow{w}\xrightarrow{\sigma}{}^{k} P'$, *for* $k \in \mathbb{N}$, *then* $\exists P''.\, P \xrightarrow{\sigma}{}^{k}\xrightarrow{w} P''$ *and* $P' \sqsupseteq\!\!\!\sim_{mt} P''$.

Intuitively, the commutation lemma states that a delay, i.e., one or more clock transitions, after a given sequence of transitions can also be made before this sequence. Moreover, the earlier a delay is performed, the slower the resulting process is. In the sequel we are mainly interested in Part (2) of the above lemma, which follows by induction on k and by employing Part (1). The proof of the simple commutation lemma is non–trivial and requires the introduction of some technical machinery. Before doing so we apply the lemma for proving the compositionality of the MT–preorder with respect to parallel composition.

Proof (Compositionality for parallel composition). According to Def. 1, it is sufficient to establish that $\mathcal{R} =_{\mathrm{df}} \{\langle P_1|P_2, Q_1|Q_2\rangle \mid P_1 \sqsupseteq\!\!\!\sim_{\mathrm{mt}} P_2,\, Q_1 \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q_2\}$ is an MT–relation. Let $\langle P_1|P_2, Q_1|Q_2\rangle \in \mathcal{R}$ be arbitrary.

The only interesting case involves matching a transition $P_1|P_2 \xrightarrow{\alpha} P_1'|P_2'$, for some P_1', P_2' and some α, since all conditions except Cond. (1) of Def. 1 coincide with the standard ones for temporal strong bisimulation. According to the operational rules for parallel composition we distinguish the following cases.

– $P_1 \stackrel{\alpha}{\longrightarrow} P_1'$ and $P_2' \equiv P_2$: Since $P_1 \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q_1$ we know of the existence of some Q_1', k, P_1'' such that $Q_1 \stackrel{\sigma}{\longrightarrow}^k \stackrel{\alpha}{\longrightarrow} Q_1'$, $P_1' \stackrel{\sigma}{\longrightarrow}^k P_1''$, and $P_1'' \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q_1'$. Moreover, $P_2 \stackrel{\sigma}{\longrightarrow}^k P_2''$ for some P_2'', as every process is lazy and can thus engage in arbitrary delays. Since $P_2 \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q_2$, there exists some Q_2' such that $Q_2 \stackrel{\sigma}{\longrightarrow}^k Q_2'$ and $P_2'' \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q_2'$. Hence by our operational rules and the definition of $\mathcal{R}$, (i) $P_1'|P_2' \stackrel{\sigma}{\longrightarrow}^k P_1''|P_2''$, (ii) $Q_1|Q_2 \stackrel{\sigma}{\longrightarrow}^k \stackrel{\alpha}{\longrightarrow} Q_1'|Q_2'$, (iii) $\langle P_1''|P_2'', Q_1'|Q_2'\rangle \in \mathcal{R}$, as desired.

– $P_2 \stackrel{\alpha}{\longrightarrow} P_2'$ and $P_1' \equiv P_1$: This case is analogous to the previous one since parallel composition is commutative.

– $\alpha = \tau$, $P_1 \stackrel{a}{\longrightarrow} P_1'$, $P_2 \stackrel{\overline{a}}{\longrightarrow} P_2'$, for some action $a \neq \tau$: Since $P_1 \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q_1$ we know of the existence of some Q_1', k, P_1'' such that $Q_1 \stackrel{\sigma}{\longrightarrow}^k \stackrel{a}{\longrightarrow} Q_1'$, $P_1' \stackrel{\sigma}{\longrightarrow}^k P_1''$, and $P_1'' \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q_1'$. Similarly, since $P_2 \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q_2$ we know of the existence of some Q_2', l, P_2'' such that $Q_2 \stackrel{\sigma}{\longrightarrow}^l \stackrel{\overline{a}}{\longrightarrow} Q_2'$, $P_2' \stackrel{\sigma}{\longrightarrow}^l P_2''$, and $P_2'' \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q_2'$. We distinguish the following cases:
  - $k = l$: Then, $P_1'|P_2' \stackrel{\sigma}{\longrightarrow}^k P_1''|P_2''$ and $Q_1|Q_2 \stackrel{\sigma}{\longrightarrow}^k \stackrel{\tau}{\longrightarrow} Q_1'|Q_2'$. Moreover, $\langle P_1''|P_2'', Q_1'|Q_2'\rangle \in \mathcal{R}$ by the definition of $\mathcal{R}$, as desired.
  - $k \neq l$: W.l.o.g. we assume $k > l$; the other case $k < l$ is analogous. Moreover, we refer to the process between the clock transitions and the action transition on the path $Q_2 \stackrel{\sigma}{\longrightarrow}^l \stackrel{\overline{a}}{\longrightarrow} Q_2'$ as $\hat{Q}_2$. Due to the laziness property of processes, there exists some $\hat{P}_2'', \hat{Q}_2''$ satisfying $P_2'' \stackrel{\sigma}{\longrightarrow}^{k-l} \hat{P}_2''$, $\hat{P}_2'' \sqsupseteq\!\!\!\sim_{\mathrm{mt}} \hat{Q}_2''$ and $\hat{Q}_2 \stackrel{\overline{a}}{\longrightarrow} Q_2' \stackrel{\sigma}{\longrightarrow}^{k-l} \hat{Q}_2''$. By Lemma 3(2) we know of the existence of some $\hat{Q}_2'$ such that $\hat{Q}_2 \stackrel{\sigma}{\longrightarrow}^{k-l} \stackrel{\overline{a}}{\longrightarrow} \hat{Q}_2'$ and $\hat{Q}_2'' \sqsupseteq\!\!\!\sim_{\mathrm{mt}} \hat{Q}_2'$. Hence, $P_1'|P_2' \stackrel{\sigma}{\longrightarrow}^k P_1''|\hat{P}_2''$ and $Q_1|Q_2 \stackrel{\sigma}{\longrightarrow}^k \stackrel{\tau}{\longrightarrow} Q_1'|\hat{Q}_2'$ by our operational rules, and $\langle P_1''|\hat{P}_2'', Q_1'|\hat{Q}_2'\rangle \in \mathcal{R}$ by the definition of $\mathcal{R}$. □

The remainder of this section is devoted to establishing the correctness of the commutation lemma. While this exercise is quite technical, it sheds some light on the nature of faster–than preorders in the context of lower time bounds. We first define a simple syntactic faster–than relation on process terms that essentially encodes the syntactic implications of our intuition that any term P should be faster than $\sigma.P$.

Definition 4. The relation $\succ \subseteq \widehat{\mathcal{P}} \times \widehat{\mathcal{P}}$ is defined as the smallest relation satisfying the following properties, for all $P, P', Q, Q' \in \widehat{\mathcal{P}}$.

Always:	(1) $P \succ P$	(2) $P \succ \sigma.P$
If $P' \succ P$ and $Q' \succ Q$:	(3) $P'\vert Q' \succ P\vert Q$	(4) $P' + Q' \succ P + Q$
	(5) $P' \setminus L \succ P \setminus L$	(6) $P'[f] \succ P[f]$
If $P' \succ P$ and x guarded in P:	(7) $P'[\mu x.\, P/x] \succ \mu x.\, P$	

Note that relation $\succ$ is not transitive and that it is also defined for open terms. It is interesting to note that $\succ$ is adopted from [17], where we studied bisimulation–based faster–than relations in the context of *upper* time bounds. The syntactic and semantic properties of $\succ$, relative to the process calculus TACS$^{\text{LT}}$ considered in this paper, are summarized in the following lemma.

Lemma 5. *Let $P, P', Q, R \in \widehat{\mathcal{P}}$, $y \in \mathcal{V}$, and $\alpha \in \mathcal{A}$. Then:*

1. *$P \succ Q$ implies $P[R/y] \succ Q[R/y]$.*
2. *$P \xrightarrow{\sigma} P'$ implies $P' \succ P$.*
3. *$Q \succ P$ and $P \xrightarrow{\alpha} P'$ implies $\exists Q'.\, Q \xrightarrow{\alpha} Q'$ and $Q' \succ P'$.*
4. *$Q \succ P$ and $P \xrightarrow{\sigma} R$ implies $R \succ Q$.*
5. *$\succ_{|\,\mathcal{P}\times\mathcal{P}}$ is an MT–relation, whence $\succ_{|\,\mathcal{P}\times\mathcal{P}} \subseteq \sqsupseteq\!\!\!\!\sim_{mt}$.*

The most important part of this lemma is Part (5). If a process is syntactically faster than another according to $\succ$, then it is also semantically faster according to $\sqsupseteq\!\!\!\!\sim_{\mathrm{mt}}$. In this light, Part (2) shows that delaying processes indeed results in faster processes.

Proof. • *Part (1):* This statement can be proved by induction on the inference length of $P \succ Q$, exactly as in [17]. The only interesting case concerns Case (7) of Def. 4, where we can assume $y \neq x$ since x is neither free in $P[\mu x.Q/x]$ nor in $\mu x.Q$, as well as $P[\mu x.Q/x] \succ \mu x.Q$ due to $P \succ Q$. Moreover, by Barendregt's Assumption, let us assume that there is no free occurrence of x in R. The induction hypothesis yields $P[R/y] \succ Q[R/y]$, whence $(P[\mu x.Q/x])[R/y] \equiv (P[R/y])[\mu x.(Q[R/y])/x] \succ \mu x.(Q[R/y]) \equiv (\mu x.Q)[R/y]$.

• *Part (2):* The proof of this statement is a straightforward induction on the structure of P.

• *Part (3):* The proof is by induction on the inference length of $P \succ Q$. The only interesting case concerns again Case (7) of Def. 4; note that Case (2) of Def. 4 is not applicable. Assume, $P' \succ P$ and $P \xrightarrow{\alpha} \hat{P}$ for some $\hat{P}$. Then we have $\mu x.P \xrightarrow{\alpha} \hat{P}[\mu x.P/x]$. By induction hypothesis, $P' \xrightarrow{\alpha} \hat{P}'$ for some $\hat{P}' \succ \hat{P}$. Hence, $P'[\mu x.P/x] \xrightarrow{\alpha} \hat{P}'[\mu x.P/x]$ and, by Part (1), $\hat{P}'[\mu x.P/x] \succ \hat{P}[\mu x.P/x]$.

• *Part (4):* The proof is again by induction on the inference length of $Q \succ P$. Note that Case (1) of Def. 4 is dealt with by Part (2). We only consider here Case (7) of Def. 4. Assume $P' \succ P$ and $P \xrightarrow{\sigma} \hat{P}$ for some $\hat{P}$. Then we have $\mu x.P \xrightarrow{\sigma} \hat{P}[\mu x.P/x]$. By induction hypothesis, $\hat{P} \succ P'$, whence $\hat{P}[\mu x.P/x] \succ P'[\mu x.P/x]$ by Part (1).

• *Part (5):* Consider arbitrary processes P, Q such that $P \succ Q$.

– $P \xrightarrow{\alpha} P'$: Due to the laziness property of our semantics regarding processes, but not necessarily terms, we know of the existence of some process Q' such that $Q \xrightarrow{\sigma} Q'$ and, by Lemma 5(4), $Q' \succ P$. When applying Lemma 5(3) we obtain some Q'' such that $Q' \xrightarrow{\alpha} Q''$ and $Q'' \succ P'$. Since P' is a process as well, there is some P'' with $P' \xrightarrow{\sigma} P''$. Finally, by Lemma 5(4), $P'' \succ Q''$.

– $Q \xrightarrow{\alpha} Q'$: This is case is dealt with by Lemma 5(3).

– $P \xrightarrow{\sigma} P'$: Since Q is a process, there is some Q' such that $Q \xrightarrow{\sigma} Q'$ and, by Lemma 5(4), $Q' \succ P$. Consequently, we must have $P' \succ Q'$ as well, according to Lemma 5(4).

– $Q \xrightarrow{\sigma} Q'$: Lemma 5(4) immediately yields $Q' \succ P$. Since P is a process, there is some P' with $P \xrightarrow{\sigma} P'$. Hence, $P' \succ Q'$ by Lemma 5(4) again. $\square$

With these prerequisites we can now prove the commutation lemma.

Proof (of Lemma 3). • *Part (1):* Let $P, P_1, P' \in \mathcal{P}$ and $w \in (\mathcal{A} \cup \{\sigma\})^*$ such that $P \xrightarrow{w} P_1 \xrightarrow{\sigma} P'$. Because of Lemma 5(5), it is sufficient to establish the existence of some $P'', P_2 \in \mathcal{P}$ such that $P \xrightarrow{\sigma} P_2 \xrightarrow{w} P''$ and $P' \succ P''$. Since every process has a unique clock derivative, we know of the existence of some P_2 with $P \xrightarrow{\sigma} P_2$. According to Lemma 5(2), $P_2 \succ P$ holds. Further since $P \xrightarrow{w} P_1$ and because of Lemma 5(5), there exists some P'' such that $P_2 \xrightarrow{w} P''$ and $P'' \succ P_1$. Now, $P'' \succ P_1$ and $P_1 \xrightarrow{\sigma} P'$ yields $P' \succ P''$ by Lemma 5(4), as desired.

• *Part (2):* Let $P, P_1, P' \in \mathcal{P}$, $w \in (\mathcal{A} \cup \{\sigma\})^*$, and $k \in \mathbb{N}$ such that $P \xrightarrow{w} P_1 \xrightarrow{\sigma}{}^k P'$. The proof of Part (2) is by induction on k. For $k = 0$, the statement holds trivially. For $k = 1$, the statement is the one of Part (1). For the induction step, consider $P \xrightarrow{w} P_1 \xrightarrow{\sigma}{}^k P'_1 \xrightarrow{\sigma} P'$, for $k \geq 1$. By the induction hypothesis we know of the existence of some P_2, P'_2 such that $P \xrightarrow{\sigma}{}^k P_2 \xrightarrow{w} P'_2$ and $P'_1 \sqsupset\!\!\!\sim_{\mathrm{mt}} P'_2$. As the TACS$^{\text{LT}}$ semantics for processes supports laziness, P'_2 can engage in a clock transition to some P''_2, i.e., $P'_2 \xrightarrow{\sigma} P''_2$. Because of $P'_1 \sqsupset\!\!\!\sim_{\mathrm{mt}} P'_2$ as well as time determinacy, we may conclude $P' \sqsupset\!\!\!\sim_{\mathrm{mt}} P''_2$. Applying the simple commutation lemma of Part (1) to $P_2 \xrightarrow{w} P'_2 \xrightarrow{\sigma} P''_2$, we obtain some P'' such that $P_2 \xrightarrow{\sigma} \xrightarrow{w} P''$ and $P''_2 \sqsupset\!\!\!\sim_{\mathrm{mt}} P''$. Hence, $P \xrightarrow{\sigma}{}^{k+1} \xrightarrow{w} P''$ and $P' \sqsupset\!\!\!\sim_{\mathrm{mt}} P''$. $\square$

The remainder of this paper studies the MT–preorder in detail. We will justify its motivation as a faster–than relation by means of formal theorems, and we will correct and generalize several statements made by Moller and Tofts in [20] concerning its semantic theory.

4 The MT–Preorder is Fully–Abstract

While the MT–preorder is algebraically appealing due to its precongruence property, it does not necessarily seem to be a natural choice for defining a faster–than relation. As mentioned earlier, Def. 1 requires that differences in delays between processes must be accounted for within one step of matching, and hence not all the future behaviour of P' in Part 1 is considered. In the following we explore an alternative *amortized* view of faster–than, where the differences can be smoothened out over several steps. Technically, we will prove that the MT–preorder is fully–abstract with respect to this amortized preorder, which demonstrates that the MT–preorder has indeed very intuitive roots.

Definition 6 (Amortized faster–than preorder). A family $(\mathcal{R}_i)_{i \in \mathbb{N}}$ of relations over $\mathcal{P}$ is a *family of faster–than relations* if, for all $i \in \mathbb{N}$, $\langle P, Q \rangle \in \mathcal{R}_i$, and $\alpha \in \mathcal{A}$:

1. $P \xrightarrow{\alpha} P'$ implies $\exists Q', k.\, Q \xrightarrow{\sigma}{}^{k} \xrightarrow{\alpha} Q'$ and $\langle P', Q' \rangle \in \mathcal{R}_{i+k}$.
2. $Q \xrightarrow{\alpha} Q'$ implies $\exists P', k{\leq}i.\, P \xrightarrow{\sigma}{}^{k} \xrightarrow{\alpha} P'$ and $\langle P', Q' \rangle \in \mathcal{R}_{i-k}$.
3. $P \xrightarrow{\sigma} P'$ implies $\exists Q', k{\geq}0.\, k \geq 1 - i$, $Q \xrightarrow{\sigma}{}^{k} Q'$, and $\langle P', Q' \rangle \in \mathcal{R}_{i-1+k}$.
4. $Q \xrightarrow{\sigma} Q'$ implies $\exists P'$, $k{\geq}0.\, k \leq i + 1$, $P \xrightarrow{\sigma}{}^{k} P'$, and $\langle P', Q' \rangle \in \mathcal{R}_{i+1-k}$.

We write $P \sqsupseteq\!\!\!\sim_i Q$ if $\langle P, Q \rangle \in \mathcal{R}_i$ for some family of faster–than relations $(\mathcal{R}_i)_{i \in \mathbb{N}}$, and call $\sqsupseteq\!\!\!\sim_0$ the *amortized faster–than preorder.*

This definition reflects our intuition that processes that perform delays later along execution paths are faster than functionally equivalent ones that perform delays earlier; this is because the former processes are executing actions at earlier absolute times (as measured from the start of the processes) than the latter ones. Def. 6 formalizes this intuition as follows: $P \sqsupseteq\!\!\!\sim_i Q$ means that Q, or rather some predecessor of Q, has already performed i clock transitions that were not matched by P; therefore, P has a credit of i clock transitions that it might perform later without a match by Q (cf. Part (3) for $k = 0$). Any extra delays of the slower process when matching an action or clock transition of the faster process, increase credit i accordingly (cf. Parts (1) and (3) for $k > 1$). Vice versa, an action or clock transition of the slower process does not necessarily have to be matched directly by the faster one: the latter may delay up to as many clock transitions as are allowed by the current credit i (cf. Parts (2) and (4)).

Processes $P =_{\text{df}} c.a.\sigma.b.\mathbf{0} + c.a.b.\mathbf{0}$ and $Q =_{\text{df}} c.a.b.\mathbf{0}$ exhibit the difference to the MT–preorder. The family $(\mathcal{R}_i)_{i \in \mathbb{N}}$ of faster–than relations defined by $\mathcal{R}_0 =_{\text{df}} \{\langle P, Q \rangle\} \cup \{\langle R, R \rangle \mid R \in \mathcal{P}\}$, $R_1 =_{\text{df}} \{\langle a.\sigma.b.\mathbf{0}, a.b.\mathbf{0} \rangle, \langle \sigma.b.\mathbf{0}, b.\mathbf{0} \rangle, \langle b.\mathbf{0}, b.\mathbf{0} \rangle, \langle \mathbf{0}, \mathbf{0} \rangle\}$ and $\mathcal{R}_i =_{\text{df}} \emptyset$, for $i > 1$, testifies to $P \sqsupseteq\!\!\!\sim_0 Q$; note that $P \xrightarrow{c} a.\sigma.b.\mathbf{0}$ is matched by $Q \xrightarrow{\sigma} \xrightarrow{c} a.b.\mathbf{0}$. However, we do not have $P \sqsupseteq\!\!\!\sim_{\text{mt}} Q$. The step $P \xrightarrow{c} a.\sigma.b.\mathbf{0}$ could only be matched by $Q \xrightarrow{\sigma}{}^{k} \xrightarrow{c} a.b.\mathbf{0}$ for some $k \in \mathbb{N}$. Since $a.\sigma.b.\mathbf{0} \xrightarrow{\sigma}{}^{k} a.\sigma.b.\mathbf{0}$, for any k, this would require $a.\sigma.b.\mathbf{0} \sqsupseteq\!\!\!\sim_{\text{mt}} a.b.\mathbf{0}$, which is clearly wrong.

It can be shown that the amortized faster–than preorder is indeed a preorder and that $(\sqsupseteq\!\!\!\sim_i)_{i \in \mathbb{N}}$ is the (componentwise) largest family of faster–than relations. However, there is an important shortcoming: $\sqsupseteq\!\!\!\sim_0$ is not preserved under parallel composition. Consider the processes P and Q above, where $P \sqsupseteq\!\!\!\sim_0 Q$. For $R =_{\text{df}} \mu x.(\sigma.d.\mathbf{0} \mid \sigma.x)$, where d is a 'fresh' action not occurring in the sorts of P and Q, one may show that $P \mid R \not\sqsupseteq\!\!\!\sim_0 Q \mid R$. The reason for this is as follows. Transition $P \mid R \xrightarrow{c} a.\sigma.b.\mathbf{0} \mid R$ would need to be matched by a sequence of transitions $Q \mid R \xrightarrow{\sigma}{}^{k} \xrightarrow{c} a.b.\mathbf{0} \mid d.\mathbf{0} \mid \cdots \mid d.\mathbf{0} \mid R$, for some $k \in \mathbb{N}$ and k parallel components $d.\mathbf{0}$, such that $a.\sigma.b.\mathbf{0} \mid R \sqsupseteq\!\!\!\sim_k a.b.\mathbf{0} \mid d.\mathbf{0} \mid \cdots \mid d.\mathbf{0} \mid R$ would hold. Now, let the latter process engage in all d–computations of the k components $d.\mathbf{0}$. Since d is a fresh action, these can only be matched by unfolding k–times process R in $a.\sigma.b.\mathbf{0} \mid R$ and executing k clock transitions and k d–transitions. Thus, $a.\sigma.b.\mathbf{0} \mid R \sqsupseteq\!\!\!\sim_0 a.b.\mathbf{0} \mid R$ would follow necessarily, i.e., no credit remains. While the right–hand process can now engage in the sequence $a.b$, the left–hand process can only match action a, but not also action b due to the lack of credit.

To address this compositionality problem of $\sqsupseteq\!\!\!\sim_0$ we refine its definition.

Definition 7 (Amortized faster–than precongruence). A family $(\mathcal{R}_i)_{i\in\mathbb{N}}$ of relations over $\mathcal{P}$ is a *precongruence family* if, for all $i \in \mathbb{N}$, $\langle P, Q\rangle \in \mathcal{R}_i$, and $\alpha \in \mathcal{A}$:

1. $P \xrightarrow{\alpha} P'$ implies $\exists Q', k.\, Q \xrightarrow{\sigma}{}^{k} \xrightarrow{\alpha} Q'$ and $\langle P', Q'\rangle \in \mathcal{R}_{i+k}$.
2. $Q \xrightarrow{\alpha} Q'$ implies $\exists P', k \leq i.\, P \xrightarrow{\sigma}{}^{k} \xrightarrow{\alpha} P'$ and $\langle P', Q'\rangle \in \mathcal{R}_{i-k}$.
3. $P \xrightarrow{\sigma} P'$ implies (a) $i > 0$ and $\langle P', Q\rangle \in \mathcal{R}_{i-1}$, or (b) $i = 0$ and $\exists Q'.\, Q \xrightarrow{\sigma} Q'$ and $\langle P', Q'\rangle \in \mathcal{R}_i$.
4. $Q \xrightarrow{\sigma} Q'$ implies $\langle P, Q'\rangle \in \mathcal{R}_{i+1}$.

We write $P \mathrel{\underset{\sim}{\sqsupset}}_i Q$ if $\langle P, Q\rangle \in \mathcal{R}_i$ for some precongruence family $(\mathcal{R}_i)_{i\in\mathbb{N}}$ and call $\mathrel{\underset{\simeq}{\sqsupset}}_0$ the *amortized faster–than precongruence*.

One can show that this amortized faster–than precongruence is indeed a preorder and that $(\mathrel{\underset{\simeq}{\sqsupset}}_i)_{i\in\mathbb{N}}$ is the (componentwise) largest family of faster–than relations. This preorder's definition is identical to the one of the amortized faster–than preorder, with the exception that a delay of the faster process now always results in consuming any available credit, while any delay of the slower process results in increasing the credit available to the faster one. As a consequence, it is easy to see that the amortized faster–than precongruence refines the amortized faster–than preorder, i.e., $\mathrel{\underset{\simeq}{\sqsupset}}_0 \subseteq \mathrel{\underset{\sim}{\sqsupset}}_0$. This is indeed a proper inclusion, as can be seen by studying the example $c.a.\sigma.b.\mathbf{0} + c.a.b.\mathbf{0} \mathrel{\underset{\sim}{\sqsupset}}_0 c.a.b.\mathbf{0}$.

Theorem 8 (Coincidence). *The preorders* $\mathrel{\underset{\simeq}{\sqsupset}}_0$ *and* $\mathrel{\underset{\sim}{\sqsupset}}_{mt}$ *coincide.*

Proof. The inclusion $\mathrel{\underset{\simeq}{\sqsupset}}_0 \subseteq \mathrel{\underset{\sim}{\sqsupset}}_{\mathrm{mt}}$ follows immediately by the definitions of these preorders and the laziness property in TACS$^{\mathrm{LT}}$; note that any credit the faster process might gain according to Def. 7 can immediately be removed via Rule (3). For establishing the other inclusion we prove that

$$\mathcal{R}_i =_{\mathrm{df}} \{ \langle P, Q\rangle \mid \exists \hat{P}.\, P \xrightarrow{\sigma}{}^{i} \hat{P} \mathrel{\underset{\sim}{\sqsupset}}_{\mathrm{mt}} Q \}$$

is a precongruence family according to Def. 7, whence $P \mathrel{\underset{\sim}{\sqsupset}}_{\mathrm{mt}} Q$ implies $\langle P, Q\rangle \in \mathcal{R}_0$. Let $\langle P, Q\rangle \in \mathcal{R}_i$ for some i be arbitrary, i.e., $P \xrightarrow{\sigma}{}^{i} \hat{P} \mathrel{\underset{\sim}{\sqsupset}}_{\mathrm{mt}} Q$. By Def. 7 we need to distinguish the following cases.

- $P \xrightarrow{\alpha} P'$: Due to the laziness property in TACS$^{\mathrm{LT}}$, there is a unique P'' such that $P' \xrightarrow{\sigma}{}^{i} P''$. By our commutation lemma, Lemma 3(2), and by time determinacy, we obtain $\hat{P} \xrightarrow{\alpha} \hat{P}'$ for some $\hat{P}'$ such that $P'' \mathrel{\underset{\sim}{\sqsupset}}_{\mathrm{mt}} \hat{P}'$. Applying Def. 1(1) to $\hat{P} \mathrel{\underset{\sim}{\sqsupset}}_{\mathrm{mt}} Q$ yields $Q', k, \hat{P}''$ satisfying $Q \xrightarrow{\sigma}{}^{k} \xrightarrow{\alpha} Q'$, $\hat{P}' \xrightarrow{\sigma}{}^{k} \hat{P}''$, and $\hat{P}'' \mathrel{\underset{\sim}{\sqsupset}}_{\mathrm{mt}} Q'$. Now, repeatedly applying Def. 1(4) to $P'' \mathrel{\underset{\sim}{\sqsupset}}_{\mathrm{mt}} \hat{P}'$, proves the existence of some P''' such that $P'' \xrightarrow{\sigma}{}^{k} P'''$ and $P''' \mathrel{\underset{\sim}{\sqsupset}}_{\mathrm{mt}} \hat{P}''$. Hence, $P' \xrightarrow{\sigma}{}^{i+k} P''' \mathrel{\underset{\sim}{\sqsupset}}_{\mathrm{mt}} Q'$, i.e., $\langle P', Q'\rangle \in \mathcal{R}_{i+k}$.
- $Q \xrightarrow{\alpha} Q'$: We know by Def. 1(2) of some $\hat{P}'$ such that $\hat{P} \xrightarrow{\alpha} \hat{P}'$ and $\hat{P}' \mathrel{\underset{\sim}{\sqsupset}}_{\mathrm{mt}} Q'$. Hence, $P \xrightarrow{\sigma}{}^{i} \xrightarrow{\alpha} \hat{P}'$ and $\langle \hat{P}', Q'\rangle \in \mathcal{R}_0$.

- $P \xrightarrow{\sigma} P'$: If $i > 0$, then we obtain $\langle P', Q\rangle \in \mathcal{R}_{i-1}$ immediately. Otherwise ($i = 0$), $P \equiv \hat{P}$, i.e., $P \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q$. Hence the existence of some Q' such that $Q \xrightarrow{\sigma} Q'$ and $P' \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q'$. This implies $\langle P', Q'\rangle \in \mathcal{R}_0$, as desired.
- $Q \xrightarrow{\sigma} Q'$: Since $\hat{P} \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q$, there exists some $\hat{P}'$ satisfying $\hat{P} \xrightarrow{\sigma} \hat{P}'$ and $\hat{P}' \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q'$. Hence, $P \xrightarrow{\sigma}{}^{i} \xrightarrow{\sigma} \hat{P}' \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q'$, i.e., $\langle P, Q'\rangle \in \mathcal{R}_{i+1}$. □

Consequently, $\sqsupseteq\!\!\!\simeq_0$ is not only a preorder but indeed a precongruence, since $\sqsupseteq\!\!\!\sim_{\mathrm{mt}}$ is a precongruence. Note, however, that the relations $\sqsupseteq\!\!\!\simeq_i$, for $i > 0$, are not precongruences; for example, $\sigma.b.\mathbf{0} \sqsupseteq\!\!\!\simeq_1 b.\mathbf{0}$ but not $a.\sigma.b.\mathbf{0} \sqsupseteq\!\!\!\simeq_1 a.b.\mathbf{0}$ due to Def. 7(3).

Theorem 9 (Full abstraction). *The preorder* $\sqsupseteq\!\!\!\simeq_0$ *is the* largest precongruence *contained in* $\sqsupseteq\!\!\!\sim_0$.

Proof. We know by universal algebra that there exists a largest precongruence $\sqsupseteq\!\!\!\sim_0^+$ contained in $\sqsupseteq\!\!\!\sim_0$, which is characterized by $\sqsupseteq\!\!\!\sim_0^+ = \{\langle P, Q\rangle \,|\, \forall\text{contexts}\, C[\_].\ C[P] \sqsupseteq\!\!\!\sim_0 C[Q]\,\}$. Because of Thm. 8, it suffices to prove $\sqsupseteq\!\!\!\sim_{\mathrm{mt}} = \sqsupseteq\!\!\!\sim_0^+$.

We have already established that $\sqsupseteq\!\!\!\sim_{\mathrm{mt}}$ is a precongruence and, by Thm. 8, that $\sqsupseteq\!\!\!\sim_{\mathrm{mt}} \;=\; \sqsupseteq\!\!\!\simeq_0 \;\subseteq\; \sqsupseteq\!\!\!\sim_0$. Hence, $\sqsupseteq\!\!\!\sim_{\mathrm{mt}} \;=\; \sqsupseteq\!\!\!\sim_{\mathrm{mt}}^+ \;\subseteq\; \sqsupseteq\!\!\!\sim_0^+$. For proving the reverse inclusion $\sqsupseteq\!\!\!\sim_0^+ \subseteq \sqsupseteq\!\!\!\sim_{\mathrm{mt}}$, it turns out to be convenient to define yet another characterization $\sqsupseteq\!\!\!\sim_{\mathrm{mt}'}$ of $\sqsupseteq\!\!\!\sim_{\mathrm{mt}}$ and prove $\sqsupseteq\!\!\!\sim_0^+ \subseteq \sqsupseteq\!\!\!\sim_{\mathrm{mt}'}$.

The preorder $\sqsupseteq\!\!\!\sim_{\mathrm{mt}'}$ is defined as $\sqsupseteq\!\!\!\sim_{\mathrm{mt}}$ when replacing Part (3) in Def. 1 by

$$\text{(3')}\quad P \xrightarrow{\sigma} P' \ \text{ implies } \exists Q', P'', k{\geq}1.\ Q \xrightarrow{\sigma}{}^{k} Q',\ P' \xrightarrow{\sigma}{}^{k-1} P'',\ \langle P'', Q'\rangle \in \mathcal{R}\,.$$

This leads to a notion of MT'–relation. First, observe that $\sqsupseteq\!\!\!\sim_{\mathrm{mt}'} \;=\; \sqsupseteq\!\!\!\sim_{\mathrm{mt}}$. The inclusion "$\supseteq$" is trivial since (3') is less restrictive than (3). For proving inclusion "$\subseteq$" we show that $\sqsupseteq\!\!\!\sim_{\mathrm{mt}'}$ is an MT–relation. This is trivial except for the case $P \xrightarrow{\sigma} P'$. In that case, due to the laziness property of TACS$^{\mathrm{LT}}$, there is a process Q' with $Q \xrightarrow{\sigma} Q'$. According to Part (4), $P' \sqsupseteq\!\!\!\sim_{\mathrm{mt}'} Q'$, as desired.

We may now establish the remaining inclusion $\sqsupseteq\!\!\!\sim_0^+ \subseteq \sqsupseteq\!\!\!\sim_{\mathrm{mt}'}$ by showing that

$$\mathcal{R}_a =_{\mathrm{df}} \{\, \langle P, Q\rangle \mid C[P] \sqsupseteq\!\!\!\sim_0 C[Q]\,\}$$

is an MT'–relation, where $C[\_] =_{\mathrm{df}} \_ \,|\, \mu x.(\sigma.d.\mathbf{0} \,|\, \sigma.x)$ for some 'fresh' action d that is not in the sorts of P and Q. Let $\langle P, Q\rangle \in \mathcal{R}_a$; according to the variation of Def. 1 we distinguish the following cases.

- $P \xrightarrow{\alpha} P'$: Hence, $C[P] \xrightarrow{\alpha} C[P']$ by the operational rules for TACS$^{\mathrm{LT}}$. Since $C[P] \sqsupseteq\!\!\!\sim_0 C[Q]$ and by the definition of $C[\_]$ we know of the existence of some Q', k such that

$$C[Q] \xrightarrow{\sigma}{}^{k} \xrightarrow{\alpha} C[Q'] \,|\, \underbrace{d.\mathbf{0} \,|\, \ldots \,|\, d.\mathbf{0}}_{k \text{ times}},$$

where $Q \xrightarrow{\sigma}{}^{k} \xrightarrow{\alpha} Q'$ and $C[P'] \sqsupseteq\!\!\!\sim_k C[Q'] \,|\, d.\mathbf{0} \,|\, \ldots \,|\, d.\mathbf{0}$. Further, consider $C[Q'] \,|\, d.\mathbf{0} \,|\, \ldots \,|\, d.\mathbf{0} \xrightarrow{d}{}^{k} C[Q']$. (The latter process is really $C[Q'] \,|\, \mathbf{0} \,|\, \ldots \,|\, \mathbf{0}$

but $\mathbf{0}$ as a parallel component never makes a difference regarding all semantic preorders considered in this paper; hence, we freely omit such components.) These action transitions must be matched by the faster process using exactly k clock transitions (according to the definition of $C[\_]$ at least k and according to Part (2) at most k clock transitions). Hence, there exist processes $P_0, P_1, P_2, \ldots, P_k$ and numbers $j_1, j_2, \ldots, j_k$ with $\sum_{1 \leq i \leq k} j_i = k$ such that, for $1 \leq i \leq k$, (i) $P_0 \equiv P'$, (ii) $C[P_{i-1}] \xrightarrow{\sigma}^{j_i} \xrightarrow{d} C[P_i]$, where $P_{i-1} \xrightarrow{\sigma}^{j_i} P_i$, and (iii) $C[P_k] \sqsupseteq\!\!\!\sim_0 C[Q']$. Thus, $\langle P_k, Q' \rangle \in \mathcal{R}_a$, as desired.

– $Q \xrightarrow{\alpha} Q'$: By TACS$^{\text{LT}}$ semantics, $C[Q] \xrightarrow{\alpha} C[Q']$. Further, by Def. 6 and due to the definition of $C[\_]$, there exists some P' such that $C[P] \xrightarrow{\alpha} C[P']$, where $P \xrightarrow{\alpha} P'$ and $C[P'] \sqsupseteq\!\!\!\sim_0 C[Q']$. Hence, $\langle P', Q' \rangle \in \mathcal{R}_a$.

– $P \xrightarrow{\sigma} P'$: Then, $C[P] \xrightarrow{\sigma} C[P'] \,|\, d.\mathbf{0}$ by our operational rules. Because of $C[P] \sqsupseteq\!\!\!\sim_0 C[Q]$ and the definition of $C[\_]$, there exists some Q' and some $k \geq 1$ such that $C[Q] \xrightarrow{\sigma}^{k} C[Q'] \,|\, d.\mathbf{0} \,|\, \cdots \,|\, d.\mathbf{0}$ with k parallel components $d.\mathbf{0}$, where $Q \xrightarrow{\sigma}^{k} Q'$, and $C[P'] \,|\, d.\mathbf{0} \sqsupseteq\!\!\!\sim_{k-1} C[Q'] \,|\, d.\mathbf{0} \,|\, \cdots \,|\, d.\mathbf{0}$. Because of the derivation $C[Q'] \,|\, d.\mathbf{0} \,|\, \cdots \,|\, d.\mathbf{0} \xrightarrow{d}^{k} C[Q']$ and since d is a fresh action not in the sort of P, we conclude that $C[P'] \,|\, d.\mathbf{0}$ performs at least (cf. definition of $C[\_]$) and at most (cf. Part (2)) $k-1$ clock transitions and k d–transitions, giving $C[P''] \sqsupseteq\!\!\!\sim_0 C[Q']$ for a process P'' satisfying $P' \xrightarrow{\sigma}^{k-1} P''$. Hence, $\langle P'', Q' \rangle \in \mathcal{R}_a$, i.e., Part (3') of the definition of $\sqsupseteq\!\!\!\sim_{\text{mt}}$ holds.

– $Q \xrightarrow{\sigma} Q'$: Here, we may derive $C[Q] \xrightarrow{\sigma} C[Q'] \,|\, d.\mathbf{0}$ and one of the following cases holds.
  - $k = 1$, i.e., $C[P] \xrightarrow{\sigma} C[P'] \,|\, d.\mathbf{0} \sqsupseteq\!\!\!\sim_0 C[Q'] \,|\, d.\mathbf{0}$: The d–transition of process $C[Q'] \,|\, d.\mathbf{0}$ must be matched by the d–transition of $C[P'] \,|\, d.\mathbf{0}$ such that $C[P'] \sqsupseteq\!\!\!\sim_0 C[Q']$.
  - $k = 0$, i.e., $C[P] \sqsupseteq\!\!\!\sim_1 C[Q'] \,|\, d.\mathbf{0}$: Here, the d–transition of $C[Q'] \,|\, d.\mathbf{0}$ can only be matched by a clock transition followed by a d–transition such that $C[P'] \sqsupseteq\!\!\!\sim_0 C[Q']$.

  In both cases we have the existence of some P' such that $P \xrightarrow{\sigma} P'$ and $\langle P', Q' \rangle \in \mathcal{R}_a$. □

Intuitively, Thms. 8 and 9 show that the MT–preorder rests on a very natural, amortized view of the notion of faster–than. Henceforth, we will call $\sqsupseteq\!\!\!\sim_{\text{mt}} = \sqsupseteq\!\!\!\sim_0$ the *strong faster–than precongruence*.

5 Axiomatizing the Moller–Tofts Preorder

We give a sound and complete axiomatization of our strong faster–than precongruence $\sqsupseteq\!\!\!\sim_{\text{mt}}$ for the class of *finite* processes, which do not contain any recursion operator. This allows one to compare our semantic theory for a calculus with lower time bounds, with the one developed for a calculus with upper time bounds presented in [17], as well as with the CCS theory of *strong bisimulation* [18].

Table 3. Axiom system for finite processes

(A1)	$t + u \;=\; u + t$	(D1)	$\mathbf{0}[f] \;=\; \mathbf{0}$	
(A2)	$t + (u + v) \;=\; (t + u) + v$	(D2)	$(\alpha.t)[f] \;=\; f(\alpha).(t[f])$	
(A3)	$t + t \;=\; t$	(D3)	$(\sigma.t)[f] \;=\; \sigma.(t[f])$	
(A4)	$t + \mathbf{0} \;=\; t$	(D4)	$(t + u)[f] \;=\; t[f] + u[f]$	
(P3)	$t + \sigma.t \;=\; t$	(C1)	$\mathbf{0} \setminus L \;=\; \mathbf{0}$	
(P4)	$\sigma.(t + u) \;=\; \sigma.t + \sigma.u$	(C2)	$(\alpha.t) \setminus L \;=\; \mathbf{0}$	$\alpha \in L \cup \overline{L}$
(P5)	$t \;\sqsupseteq\; \sigma.t$	(C3)	$(\alpha.t) \setminus L \;=\; \alpha.(t \setminus L)$	$\alpha \notin L \cup \overline{L}$
		(C4)	$(\sigma.t) \setminus L \;=\; \sigma.(t \setminus L)$	
(P6)	$\alpha.t \;=\; \alpha.\sigma.t + \alpha.t$	(C5)	$(t + u) \setminus L \;=\; (t \setminus L) + (u \setminus L)$	

Let $t \equiv \sum_{i \in I} \alpha_i.t_i \; [+\sigma.t_\sigma\,]$ and $u \equiv \sum_{j \in J} \beta_j.u_j \; [+\sigma.u_\sigma\,]$.

(E) $t|u = \sum_{i \in I} \alpha_i.(t_i|u) + \sum_{j \in J} \beta_j.(t|u_j) + \sum_{\alpha_i = \overline{\beta}_j} \tau.(t_i|u_j) +$

$$\begin{cases} \mathbf{0} & \text{if both } \sigma.t_\sigma,\ \sigma.u_\sigma \text{ are absent} \\ \sigma.(((\sum_{i \in I} \alpha_i.t_i) \,+\, t_\sigma) \,|\, (\sum_{j \in J} \beta_j.u_j)) & \text{if only } \sigma.u_\sigma \text{ is absent} \\ \sigma.((\sum_{i \in I} \alpha_i.t_i) \,|\, ((\sum_{j \in J} \beta_j.u_j) \,+\, u_\sigma)) & \text{if only } \sigma.t_\sigma \text{ is absent} \\ \sigma.(((\sum_{i \in I} \alpha_i.t_i) \,+\, t_\sigma) \,|\, ((\sum_{j \in J} \beta_j.u_j) \,+\, u_\sigma)) & \text{otherwise} \end{cases}$$

The axioms for our MT–precongruence are shown in Table 3, where a term in square brackets is optional. Moreover, $\sum$ is the indexed version of $+$, and we adopt the convention that the sum over the empty index set is identified with process $\mathbf{0}$. Any axiom of the form $t = u$ should be read as two axioms $t \sqsupseteq u$ and $u \sqsupseteq t$. We write $\vdash t \sqsupseteq u$ if $t \sqsupseteq u$ can be derived from the axioms.

Axioms (A1)–(A4), (D1)–(D4), and (C1)–(C5) are exactly the ones for strong bisimulation in CCS [18]. Hence, the semantic theory of our calculus is distinguished from the one for strong bisimulation by the additional Axioms (P3)–(P6) and the refined expansion law (E). Further, it is distinguished from the one for the faster–than preorder for upper time bounds [17] by leaving out Axioms (P1) and (P2) related to enforcing upper time bounds, and by adding Axiom (P6). Intuitively, this added axiom states that inserting a delay within a path of a process does not alter the speed of the process, as long as there exists a functionally equivalent path without delay; this shows that our theory concentrates on best–case behavior by ignoring the slower summand that has the optional delay. Axiom (P6) generalizes to

$$\text{(P6')} \qquad \alpha.P \;=\; \alpha.\sigma^k.P + \alpha.P\,,$$

for any $k \in \mathbb{N}$, by repeated application; here, "σ^k." stands for k nested clock prefixes. Axiom (P3) is similar in spirit to Axiom (P6) but cannot be derived from the other axioms. Axiom (P4) is a standard axiom in timed process algebras and testifies to the fact that time is a deterministic concept and does not resolve choices. Finally, Axiom (P5) encodes our elementary intuition of clock prefixes and speed within TACS$^{\text{LT}}$, namely that any process t is faster than process $\sigma.t$, which *must* delay the execution of t by one clock tick.

The correctness of our axioms relative to $\gtrsim_{\text{mt}}$ can be established as usual [18]. Note that all axioms, with the exception of the Expansion Axiom (E) and Axiom (P3), are sound for arbitrary processes, not only for finite ones. For example, the correctness of Axiom (P5) follows from our syntactic relation $\succ$ and Lemma 5(5). The correctness of direction "$\alpha.t \sqsupseteq \alpha.\sigma.t + \alpha.t$" of Axiom (P6) is due to the correctness of Axioms (P5) and (A3). For proving the correctness of direction "$\alpha.\sigma.t + \alpha.t \sqsupseteq \alpha.t$", the only interesting case is the matching of $\alpha.\sigma.t + \alpha.t \xrightarrow{\alpha} \sigma.t$. Here, we consider $\alpha.t \xrightarrow{\sigma} \alpha.t \xrightarrow{\alpha} t$, and observe $\sigma.t \xrightarrow{\sigma} t$ and $t \sqsupseteq\!\!\!\sim_{\mathrm{mt}} t$. It should be noted here that the axioms presented in [20] do not completely correspond with the MT–preorder, as has also been noted by Moller and Tofts since the publication of their paper in 1991 [*priv. commun.*]. For example, $a.\sigma.b.\mathbf{0} + a.b.\mathbf{0}$ is as fast as $a.b.\mathbf{0}$, which does not seem to be derivable from the axioms in [20]. In our theory, this example is a simple instantiation of Axiom (P6).

The only correctness proofs we provide in more detail concern the expansion axiom and Axiom (P3). Moller and Tofts claim in [20] that the "standard" expansion law [18] for faster–than relations based on lower time bounds does not hold, even for finite processes. While this observation is true for arbitrary processes, it is incorrect for finite ones. As a simple example we have $a.\mathbf{0} \,|\, \sigma.b.\mathbf{0} = a.(\mathbf{0}|\sigma.b.\mathbf{0}) + \sigma.(a.\mathbf{0}|b.\mathbf{0})$, contrary to the claims in [20].

Proof (Correctness of Axiom (E) for finite processes). It suffices to consider the case $P \equiv \sum_{i\in I} \alpha_i.P_i + \sigma.P_\sigma$ and $Q \equiv \sum_{j\in J} \beta_j.Q_j + \sigma.Q_\sigma$. The other three cases are similar: in fact, the first case is obvious, while the second and third case can be derived from the fourth by considering $u_\sigma \equiv \mathbf{0}$ and $t_\sigma \equiv \mathbf{0}$, respectively. For notational convenience we simply abbreviate $\sum_{i\in I} \alpha_i.P_i$ by $\sum_i$ and $\sum_{j\in J} \beta_j.Q_j$ by $\sum_j$. To prove the expansion axiom correct, we show that (i) $\mathcal{R} \cup \{\langle P|Q, \sum_{i\in I} \alpha_i.(P_i|Q) + \sum_{j\in J} \beta_j.(P|Q_j) + \sum_{\alpha_i=\overline{\beta}_j} \tau.(P_i|Q_j) + \sigma.(\sum_i + P_\sigma | \sum_j + Q_\sigma)\rangle\} \cup \sqsupseteq\!\!\!\sim_{\mathrm{mt}}$ and (ii) $\mathcal{R}^{-1} \cup \{\langle \sum_{i\in I} \alpha_i.(P_i|Q) + \sum_{j\in J} \beta_j.(P|Q_j) + \sum_{\alpha_i=\overline{\beta}_j} \tau.(P_i|Q_j) + \sigma.(\sum_i + P_\sigma | \sum_j + Q_\sigma)\,,\, P|Q\rangle\} \cup \sqsupseteq\!\!\!\sim_{\mathrm{mt}}$ are MT–relations, where

$$
\begin{aligned}
\mathcal{R} &= \{\, \langle \mathit{lhs}, \mathit{rhs}\rangle \mid \exists k \in \mathbb{N}.\ P_\sigma \xrightarrow{\sigma}{}^{k} P'_\sigma \text{ and } Q_\sigma \xrightarrow{\sigma}{}^{k} Q'_\sigma \,\}\\
\mathit{lhs} &= (\sum_i + P'_\sigma) \,|\, (\sum_j + Q'_\sigma)\\
\mathit{rhs} &= \sum_{i\in I} \alpha_i.(P_i|Q) + \sum_{j\in J} \beta_j.(P|Q_j) + \sum_{\alpha_i=\overline{\beta}_j} \tau.(P_i|Q_j) + ((\sum_i + P'_\sigma) \,|\, (\sum_j + Q'_\sigma))
\end{aligned}
$$

We also implicitly exploit the correctness of Axiom (P6). Obviously, the action transitions of the left–hand side and of the right–hand side of Axiom (E) match, while the matching of a clock transition leads to the pair in $\mathcal{R}$ for $k = 0$. Hence, it is sufficient to consider some arbitrary pair $\langle \mathit{lhs}, \mathit{rhs}\rangle \in \mathcal{R}$. Thus, there exists some $k \in \mathbb{N}$ such that $P_\sigma \xrightarrow{\sigma}{}^{k} P'_\sigma$ and $Q_\sigma \xrightarrow{\sigma}{}^{k} Q'_\sigma$.

For the proof of Claim (i), note that action transitions of *lhs* are trivially matched by *rhs*. The converse is also the case for most action transitions, except $\mathit{rhs} \xrightarrow{\alpha_i} P_i \,|\, (\sum_{j\in J} \beta_j.Q_j + \sigma.Q_\sigma)$, which deserve a closer look. These transitions can be matched by $\mathit{lhs} \xrightarrow{\alpha_i} P_i \,|\, (\sum_{j\in J} \beta_j.Q_j + Q'_\sigma)$. Since $Q_\sigma \xrightarrow{\sigma}{}^{k} Q'_\sigma$ we know by Lemma 5 and the correctness of Axiom (P5) that $Q'_\sigma \sqsupseteq\!\!\!\sim_{\mathrm{mt}} Q_\sigma \sqsupseteq\!\!\!\sim_{\mathrm{mt}} \sigma.Q_\sigma$. Thus, $P_i \,|\, (\sum_{j\in J} \beta_j.Q_j + Q'_\sigma) \sqsupseteq\!\!\!\sim_{\mathrm{mt}} P_i \,|\, (\sum_{j\in J} \beta_j.Q_j + \sigma.Q_\sigma)$. Further, consider the clock transitions of *lhs* and *rhs*, i.e., $\mathit{lhs} \xrightarrow{\sigma} \mathit{lhs}' \equiv (\sum_i +P''_\sigma) \,|\, (\sum_j +Q''_\sigma)$ and $\mathit{rhs} \xrightarrow{\sigma} \mathit{rhs}' \equiv \sum_{i\in I} \alpha_i.(P_i|Q) + \sum_{j\in J} \beta_j.(P|Q_j) + \sum_{\alpha_i=\overline{\beta}_j} \tau.(P_i|Q_j) + (\sum_i +P''_\sigma \,|\, \sum_j +Q''_\sigma)$, for P''_σ, Q''_σ satisfying $P'_\sigma \xrightarrow{\sigma} P''_\sigma$ and $Q'_\sigma \xrightarrow{\sigma} Q''_\sigma$. Since $P_\sigma \xrightarrow{\sigma}{}^k P'_\sigma \xrightarrow{\sigma} P''_\sigma$ and $Q_\sigma \xrightarrow{\sigma}{}^k Q'_\sigma \xrightarrow{\sigma} Q''_\sigma$, we have $\langle \mathit{lhs}', \mathit{rhs}' \rangle \in \mathcal{R}$.

For the proof of Claim (ii), the following property is essential:

$$\forall P \in \mathcal{P}_{\mathsf{fin}}.\, \exists n \in \mathbb{N}.\, \forall P', P'' \in \mathcal{P}_{\mathsf{fin}}.\; P \xrightarrow{\sigma}{}^n P',\; P' \xrightarrow{\sigma} P'' \text{ implies } P' \equiv P'' \qquad (*)$$

This property can be proved by induction on the structure of finite processes. When proving Claim (ii) observe that clock transitions and most action transitions can be dealt with as before. The interesting part of the proof is the matching of action transitions of the form $\mathit{rhs} \xrightarrow{\alpha_i} P_i \,|\, (\sum_{j\in J} \beta_j.Q_j + \sigma.Q_\sigma)$. We consider the transition sequence $\mathit{lhs} \xrightarrow{\sigma}{}^{\max+1} \xrightarrow{\alpha_i} P_i \,|\, (\sum_{j\in J} \beta_j.Q_j + \hat{Q}_\sigma)$, where max is the maximal number of clock transitions before process Q_σ starts idling, according to Property $(*)$, and where $\hat{Q}_\sigma$ is the unique process such that $Q_\sigma \xrightarrow{\sigma}{}^{\max} \hat{Q}_\sigma$. Further, $\mathit{rhs} \xrightarrow{\alpha_i} P_i \,|\, (\sum_{j\in J} \beta_j.Q_j + \sigma.Q_\sigma) \xrightarrow{\sigma}{}^{\max+1} P'_i \,|\, (\sum_{j\in J} \beta_j.Q_j + \hat{Q}_\sigma)$, where P'_i is the unique process satisfying $P_i \xrightarrow{\sigma}{}^{\max+1} P'_i$. Thus, $P'_i \sqsupseteq\!\!\!\sim_{\mathrm{mt}} P_i$ by Lemma 5 and $P'_i \,|\, (\sum_{j\in J} \beta_j.Q_j + \hat{Q}_\sigma) \sqsupseteq\!\!\!\sim_{\mathrm{mt}} P_i \,|\, (\sum_{j\in J} \beta_j.Q_j + \hat{Q}_\sigma)$ by Thm. 2, whence the expansion axiom is valid for finite processes. □

The above proof relies on Property $(*)$ that does not hold, e.g., for the recursive process $D =_{\mathrm{df}} \mu x.(d.\mathbf{0} \,|\, \sigma.x)$. When applying Axiom (E) to $a.\mathbf{0} \,|\, \sigma.D$ we obtain $a.\mathbf{0} \,|\, \sigma.D = a.(\mathbf{0} \,|\, \sigma.D) + \sigma.(a.\mathbf{0}|D)$. However, $a.(\mathbf{0} \,|\, \sigma.D) + \sigma.(a.\mathbf{0}|D) \not\sqsupseteq\!\!\!\sim_{\mathrm{mt}} a.\mathbf{0} \,|\, \sigma.D$. Assume otherwise; then, by Def. 1, the σ–derivatives of both processes must be related, i.e., $a.(\mathbf{0} \,|\, \sigma.D) + (a.\mathbf{0} \,|\, D) \sqsupseteq\!\!\!\sim_{\mathrm{mt}} a.\mathbf{0} \,|\, D$ would hold. However, if the allegedly faster process performs an a–transition to $(\mathbf{0} \,|\, \sigma.D)$, then the slower should match this after some delay of, say, $k \geq 1$ clock transitions; the case where $k = 0$ is obvious. According to Def. 1 we obtain

$$(\mathbf{0} \,|\, D \,|\, \underbrace{d.\mathbf{0} \,|\, \cdots \,|\, d.\mathbf{0}}_{k-1 \text{ times}}) \sqsupseteq\!\!\!\sim_{\mathrm{mt}} (\mathbf{0} \,|\, D \,|\, \underbrace{d.\mathbf{0} \,|\, \cdots \,|\, d.\mathbf{0}}_{k \text{ times}}).$$

This is clearly invalid, as the slower process can engage in k consecutive d–transitions of which the faster process can only match the first $k-1$ transitions. Hence, Axiom (E) is not universally correct. We leave it to future work to see how far the expansion law might be generalized.

Proof (Correctness of Axiom (P3) for finite processes). Direction "$t \sqsupseteq t + \sigma.t$" can be derived from Axioms (P5) and (A3) and is thus correct for arbitrary processes. For establishing the correctness of the reverse direction we show that

$$\mathcal{R} =_{\mathrm{df}} \{\langle P + Q, P \rangle \,|\, Q \xrightarrow{\sigma} P \text{ and } P \text{ satisfies Property } (*)\}$$

is an MT–relation. Thus, let $\langle P+Q, P\rangle \in \mathcal{R}$, and let max be the number n in Property $(*)$ for P; as mentioned earlier, every finite process satisfies this property.

The only interesting case arises when $P+Q \xrightarrow{\alpha} Q'$ due to $Q \xrightarrow{\alpha} Q'$, for some action α and finite process Q'. Because of the laziness property of TACS$^{\text{LT}}$, there exists some $\hat{Q}$ such that $Q' \xrightarrow{\sigma}^{\max+1} \hat{Q}$. We may then apply the commutation lemma, Lemma 3(2), to obtain $\hat{P}, P'$ satisfying $Q \xrightarrow{\sigma} P \xrightarrow{\sigma}^{\max} \hat{P}$, $\hat{P} \xrightarrow{\alpha} P'$, and $\hat{Q} \sqsupseteq\!\!\!\sim_{\text{mt}} P'$. Further, by the choice of max, we have $\hat{P} \xrightarrow{\sigma} \hat{P}$. Hence we have satisfied Def. 1(1): $P \xrightarrow{\sigma}^{\max+1} \hat{P} \xrightarrow{\alpha} P'$, $Q' \xrightarrow{\sigma}^{\max+1} \hat{Q}$, and $\hat{Q} \sqsupseteq\!\!\!\sim_{\text{mt}} P'$.

Direction "$\sqsupseteq$" of Axiom (P3) does not hold for arbitrary processes; for example, $D + \sigma.D \;\not\sqsupseteq\!\!\!\sim_{\text{mt}} D$ using process D from above. Consider the matching of transitions $D + \sigma.D \xrightarrow{\sigma}^2 (D \,|\, d.\mathbf{0} \,|\, d.\mathbf{0}) + (D \,|\, d.\mathbf{0}) \xrightarrow{d} D$ performed by the left–hand side, which enforces for the right–hand side process $D \xrightarrow{\sigma}^2 \xrightarrow{\sigma}^k \xrightarrow{d} D_{k+1} \equiv D \,|\, d.\mathbf{0} \,|\, \cdots \,|\, d.\mathbf{0}$ with $k+1$ components $d.\mathbf{0}$. But $D_k \;\not\sqsupseteq\!\!\!\sim_{\text{mt}} D_{k+1}$, as the former process cannot match the $k+1$ d–transitions of the latter one.

The proof for the completeness of our axiomatization is based on the following notion of *normal form*.

Definition 10 (Normal form). A finite process t is *in normal form* if

$$t \;\equiv\; \sum_{i\in I} \alpha_i.t_i \;\; [+\, \sigma.t_\sigma\,]\,,$$

where (i) I denotes a finite index set, (ii) $\alpha_i \in \mathcal{A}$ for all $i \in I$, (iii) all the t_i are in normal form, and (iv) the subterm in brackets is optional and, if it exists, t_σ is in normal form $\sum_{j\in J} \beta_j.u_j$ $[+\, \sigma.u_\sigma\,]$ and $\forall i \in I\, \exists j \in J.\ \alpha_i.t_i \equiv \beta_j.u_j$.

Observe that the unique clock derivative t' of a normal form is again in normal form; its size is not larger, and smaller if summand $\sigma.t_\sigma$ is present. Further, $\vdash t' = t_\sigma$ by Cond. (iv).

Proposition 11 (Rewriting into normal forms). *For any finite process t, there exists some finite process u in normal form such that $\vdash t = u$.*

The proof is by induction on the structure of finite processes and is quite straightforward. We only note here that Cond. (iv) of Def. 10 can be achieved by applying Axiom (P3). For proving our axiom system complete, the following technical lemmas are useful.

Lemma 12. *Let $t \equiv \sum_{i\in I} \alpha_i.t_i$ $[+\, \sigma.t_\sigma\,]$ be in normal form, and let terms t', u and $k \in \mathbb{N}$ such that $t \xrightarrow{\sigma}^k t'$ and $\vdash t' \sqsupseteq u$. Then $\vdash t \sqsupseteq \sigma^k.u$.*

Proof. The proof is by induction on k. For $k = 0$, i.e., $t' \equiv t$, the statement is trivial. For $k = 1$ we know by the operational rules for TACS$^{\text{LT}}$ that $t' \equiv \sum_{i\in I} \alpha_i.t_i$ $[+\; t_\sigma\,]$. Then, by repeated application of Axioms (P5) and (P4),

$\vdash t = \sum_{i \in I} \alpha_i.t_i \; [+ \; \sigma.t_\sigma \;] \sqsupseteq \sum_{i \in I} \sigma.\alpha_i.t_i \; [+ \; \sigma.t_\sigma \;] = \sigma.t' \sqsupseteq \sigma.u$. For $k+1 > 1$ we have $t \xrightarrow{\sigma} t'' \xrightarrow{\sigma}{}^k t'$, with t'' being in normal form. Then, by induction, $\vdash t'' \sqsupseteq \sigma^k.u$. Further $\vdash t \sqsupseteq \sigma^{k+1}.u$, according to the case for $k = 1$. □

Lemma 13. *Let $t \equiv \sum_{i \in I} \alpha_i.t_i + \sigma.t_\sigma$ be in normal form and γ, t', k such that $t \xrightarrow{\sigma}{}^k \xrightarrow{\gamma} t'$. Then, there exists a sub–term $\gamma.t'$ of t with $\vdash t = t + \sigma^k.\gamma.t'$.*

Proof. The statement is trivial for $k = 0$. If $k > 0$ we proceed by induction on k. For the induction base, $k = 1$, we have $t \xrightarrow{\sigma} t''$ for $t'' \equiv \sum_{i \in I} \alpha_i.t_i + t_\sigma$, where $t_\sigma \equiv \sum_{j \in J} \beta_j.u_j \; [+ \; \sigma.u_\sigma \;]$ satisfying $\forall i \in I \, \exists j \in J. \; \alpha_i.t_i \equiv \beta_j.u_j$ by Def. 10(iv). Hence, $\gamma.t' \equiv \beta_j.u_j$, for some $j \in J$. The desired property then holds simply by applying Axioms (A3) and (P4). Regarding the induction step, recall that the unique σ–derivative t'' of t is itself in normal form; a subterm $\gamma.t'$ of t'' is also a subterm of t and obviously $\vdash t'' = t_\sigma$. Then, $\vdash t = t + \sigma.t_\sigma = t + \sigma.t'' = t + \sigma.(t'' + \sigma^k.\gamma.t') = t + \sigma^{k+1}.\gamma.t'$, as desired, where the third equality holds by induction hypothesis. □

We are now able to state and prove the main result of this section.

Theorem 14 (Correctness & completeness). *For finite processes t and u we have: $\vdash t \sqsupseteq u$ if and only if $t \gtrsim_{mt} u$.*

Proof. The correctness "$\Longrightarrow$" of our axiom system follows by induction on the length of the inference $\vdash t \sqsupseteq u$, as usual. We concentrate on proving completeness "$\Longleftarrow$". By Prop. 11 is suffices to prove this implication for processes t and u in normal form, i.e., $t \equiv \sum_{i \in I} \alpha_i.t_i \;\; [+ \; \sigma.t_\sigma \;]$ and $u \equiv \sum_{j \in J} \beta_i.u_i \;\; [+ \; \sigma.u_\sigma \;]$. We proceed by induction on the sum of the process sizes of t and u. If this sum is zero we have $t \equiv u \equiv \mathbf{0}$, and we are done. Otherwise, we consider four cases, depending on whether each of the optional σ–summands $\sigma.t_\sigma$ and $\sigma.u_\sigma$ is present.

– Both summands $\sigma.t_\sigma$ and $\sigma.u_\sigma$ are absent: Hence, $t \equiv \sum_{i \in I} \alpha_i.t_i$ and $u \equiv \sum_{j \in J} \beta_j.u_j$. Due to $t \gtrsim_{\mathrm{mt}} u$ we may derive a couple of important properties:

  1. $\forall i \in I. \exists j \in J, \, k \in \mathbb{N}. \; \alpha_i{=}\beta_j, \, t_i \xrightarrow{\sigma}{}^k t'$ and $t' \gtrsim_{\mathrm{mt}} u_j$ (cf. Def. 1(1)). The induction hypothesis yields $\vdash t' \sqsupseteq u_j$; recall that t' is in normal form. Hence by Lemma 12, $\vdash t_i \sqsupseteq \sigma^k.u_j$ which implies $\vdash \alpha_i.t_i \sqsupseteq \beta_j.\sigma^k.u_j$. With Axiom (P6') we conclude $\vdash \beta_j.u_j + \alpha_i.t_i \sqsupseteq \beta_j.u_j + \beta_j.\sigma^k.u_j = \beta_j.u_j$.
  2. $\forall j \in J. \exists i \in I. \; \beta_j{=}\alpha_i$ and $t_i \gtrsim_{\mathrm{mt}} u_j$ (cf. Def. 1(2)). By induction hypothesis, $\vdash t_i \sqsupseteq u_j$ holds, whence $\vdash \alpha_i.t_i \sqsupseteq \beta_j.u_j$.

  We may now conclude this case as follows:

$$
\begin{array}{rl}
 & \vdash t = \sum_{i \in I} \alpha_i.t_i \\
(2, \text{A3}) & \quad \sqsupseteq \sum_{j \in J} \beta_j.u_j + \sum_{i \in I} \alpha_i.t_i \\
(1) & \quad \sqsupseteq \sum_{j \in J} \beta_j.u_j = u
\end{array}
$$

– Summand $\sigma.t_\sigma$ is present and $\sigma.u_\sigma$ absent: Because of property $\sum_{i\in I} \alpha_i.t_i + \sigma.t_\sigma \gtrsim_{\mathrm{mt}} \sum_{j\in J} \beta_j.u_j$ we may derive the following properties similar to the previous case:
  1. $\forall i \in I. \exists j \in J. \vdash \beta_j.u_j + \alpha_i.t_i \sqsupseteq \beta_j.u_j$.
  2. $\forall j \in J. \exists i \in I. \vdash \alpha_i.t_i \sqsupseteq \beta_j.u_j$.
  3. When considering initial clock transitions of t and u we obtain $t_\sigma \gtrsim_{\mathrm{mt}} u$; note Cond. (iv) of Def. 10. Since t_σ is in normal form, the induction hypothesis applies and yields $\vdash t_\sigma \sqsupseteq u$.

  We may now finish off the case.

$$\begin{array}{cl} & \vdash t = \sum_{i\in I} \alpha_i.t_i + \sigma.t_\sigma \\ (2,\ \text{A3}) & \quad \sqsupseteq \sum_{j\in J} \beta_j.u_j + \sigma.t_\sigma + \sum_{i\in I} \alpha_i.t_i \\ (3) & \quad \sqsupseteq u + \sigma.u + \sum_{i\in I} \alpha_i.t_i \\ (\text{P3}) & \quad = u + \sum_{i\in I} \alpha_i.t_i \\ (1) & \quad \sqsupseteq u \end{array}$$

– Summand $\sigma.t_\sigma$ is absent and $\sigma.u_\sigma$ present: Here we have $\sum_{i\in I} \alpha_i.t_i \gtrsim_{\mathrm{mt}} \sum_{j\in J} \beta_j.u_j + \sigma.u_\sigma$. When considering a clock transition of both processes, Def. 1 implies $t \gtrsim_{\mathrm{mt}} \sum_{j\in J} \beta_j.u_j + u_\sigma$. As the right–hand side process is a normal form of smaller size than the one of u we may apply the induction hypothesis and Axiom (P5) to obtain $\vdash t \sqsupseteq \sum_{j\in J} \beta_j.u_j + u_\sigma \sqsupseteq \sum_{j\in J} \beta_j.u_j + \sigma.u_\sigma = u$.

– Both summands $\sigma.t_\sigma$ and $\sigma.u_\sigma$ are present: By the premise $\sum_{i\in I} \alpha_i.t_i + \sigma.t_\sigma \gtrsim_{\mathrm{mt}} \sum_{j\in J} \beta_j.u_j + \sigma.u_\sigma$ we may conclude the validity of the following properties, similar to the previous cases:
  1. By Def. 1(1) and induction hypothesis we have $\forall i \in I. \exists u', k.\ u \xrightarrow{\sigma}{}^k \xrightarrow{\alpha_i} u'$, $t_i \xrightarrow{\sigma}{}^k t'$, and $\vdash t' \sqsupseteq u'$. Lemma 12 yields $\vdash t_i \sqsupseteq \sigma^k.u'$, whence $\vdash \alpha_i.t_i \sqsupseteq \alpha_i.\sigma^k.u' \sqsupseteq \sigma^k.\alpha_i.\sigma^k.u'$ by Axiom (P5). We may now apply this to $\vdash u = u + \sigma^k.\alpha_i.\sigma^k.u'$, which follows from Lemma 13 by applying Axiom (P6) k–times, in order to obtain $\vdash u + \alpha_i.t_i \sqsupseteq u$.
  2. $\forall j \in J. \exists i \in I. \vdash \alpha_i.t_i \sqsupseteq \beta_j.u_j$.
  3. $\vdash t_\sigma \sqsupseteq u_\sigma$ (cf. Def. 1, Parts (3) and (4)).

  We may now finish this case as follows.

$$\begin{array}{cl} (2,\ \text{A3}) & \vdash t \sqsupseteq \sum_{j\in J} \beta_j.u_j + t \\ (3) & \quad \sqsupseteq u + \sum_{i\in I} \alpha_i.t_i \\ (1) & \quad \sqsupseteq u \end{array}$$

This completes the proof of Thm. 14. □

6 Example

This section applies our semantic theory to a simple example dealing with two implementations of a two–place storage in terms of *two cells* and a *buffer*, respectively (cf., [18]). For simplifying the presentation we specify recursion via recursive process equations in the style of Milner [18], instead of using our recursion operator. The *two–cells* system is defined as the parallel composition of two one–place cells $C_0 \stackrel{\text{def}}{=} in.C_1$, where $C_1 \stackrel{\text{def}}{=} \sigma.\overline{out}.C_0$. The two–place buffer B_0 is given by the process equations $B_0 \stackrel{\text{def}}{=} in.B_1$, $B_1 \stackrel{\text{def}}{=} \sigma.\overline{out}.B_0 + in.B_2$ and $B_2 \stackrel{\text{def}}{=} \sigma.\overline{out}.B_1$. As is reflected by the σ–prefixes in front of the $\overline{out}$–prefixes, both cells C_0 and the two–place buffer B_0 have to delay at least one time unit until they can offer a communication on port $\overline{out}$. Intuitively, one would expect the two cell system to be strictly faster, since if both cells are full, then both data items stored may be output after a delay of only one time unit, while the buffer requires a delay of at least two time units until it may release the second data item.

As desired, our semantic theory for TACS$^{\text{LT}}$ relates $C_0 \,|\, C_0$ and B_0. Formally, this may be witnessed by the MT–relation given below, in which we employ the abbreviations $C_1' =_{\text{df}} \overline{out}.C_0$, $B_1' =_{\text{df}} \overline{out}.B_0 + in.B_2$, and $B_2' =_{\text{df}} \overline{out}.B_1$.

$$\begin{array}{llll}
\langle C_0 \,|\, C_0 \,, B_0 \rangle & \langle C_1 \,|\, C_0 \,, B_1 \rangle & \langle C_0 \,|\, C_1 \,, B_1 \rangle & \langle C_1' \,|\, C_0 \,, B_1' \rangle \\
\langle C_0 \,|\, C_1' \,, B_1' \rangle & \langle C_1 \,|\, C_1 \,, B_2 \rangle & \langle C_1' \,|\, C_1 \,, B_2 \rangle & \langle C_1 \,|\, C_1' \,, B_2 \rangle \\
\langle C_1' \,|\, C_1' \,, B_2' \rangle & \langle C_1' \,|\, C_0 \,, B_1 \rangle & \langle C_0 \,|\, C_1' \,, B_1 \rangle &
\end{array}$$

It is easy to check, by referring to Def. 1, that this relation is indeed an MT–relation, whence $C_0 \,|\, C_0 \gtrsim_{\text{mt}} B_0$. Vice versa, $B_0 \gtrsim_{\text{mt}} C_0 \,|\, C_0$ does *not* hold according to Def. 1, since $C_0 \,|\, C_0$ can engage in the transition sequence $C_0 \,|\, C_0 \xrightarrow{in}\xrightarrow{in}\xrightarrow{\sigma}\xrightarrow{\overline{out}}\xrightarrow{\overline{out}}$, which cannot be matched by B_0. Thus, the two–cells system is faster than the two–place buffer in all contexts, although functionally equivalent, which matches our intuition mentioned above.

Another example, which compares the speeds of different forms of mail delivery and originates in [20], can be adapted from our earlier work on faster–than relations for processes with upper time bounds [17]. This adaptation only requires one to interpret σ–prefixes as lower time bounds instead of upper time bounds. The axiomatic reasoning may then proceed as in [17], which only employs axioms that are part of the axiom system presented in Sec. 5, too.

7 Abstracting from Internal Computation

As usual in process algebra, one wishes to coarsen a semantic theory by abstracting from the internal action τ, which is supposed to be hidden from an external observer. While doing this is usually quite straightforward for CCS–based calculi [18], it turns out to be highly non–trivial here, which we guess may be the reason why it has not been attempted by Moller and Tofts in [20].

We start off by defining a weak version of our reference preorder, the amortized faster–than preorder, which requires us to introduce the following auxiliary notations. For any action α we define $\hat{\alpha} =_{\text{df}} \epsilon$, if $\alpha = \tau$, and $\hat{\alpha} =_{\text{df}} \alpha$, otherwise. Further, we let $\stackrel{\epsilon}{\Longrightarrow} =_{\text{df}} \stackrel{\tau}{\longrightarrow}^*$ and write $P \stackrel{\gamma}{\Longrightarrow} Q$, where $\gamma \in \mathcal{A} \cup \{\sigma\}$, if there exist R and S such that $P \stackrel{\epsilon}{\Longrightarrow} R \stackrel{\gamma}{\longrightarrow} S \stackrel{\epsilon}{\Longrightarrow} Q$. We also let $\stackrel{\sigma}{\Longrightarrow}^0$ stand for $\stackrel{\epsilon}{\Longrightarrow}$.

Definition 15 (Weak amortized faster–than preorder). A family $(\mathcal{R}_i)_{i \in \mathbb{N}}$ of relations over $\mathcal{P}$ is a *family of weak faster–than relations* if, for all $i \in \mathbb{N}$, $\langle P, Q \rangle \in \mathcal{R}_i$, and $\alpha \in \mathcal{A}$:

1. $P \xrightarrow{\alpha} P'$ implies $\exists Q', k, k'.\, Q \stackrel{\sigma}{\Longrightarrow}^{k} \stackrel{\hat{\alpha}}{\Longrightarrow} \stackrel{\sigma}{\Longrightarrow}^{k'} Q'$ and $\langle P', Q' \rangle \in \mathcal{R}_{i+k+k'}$.
2. $Q \xrightarrow{\alpha} Q'$ implies $\exists P', k, k'.\, k{+}k' \leq i,\ P \stackrel{\sigma}{\Longrightarrow}^{k} \stackrel{\hat{\alpha}}{\Longrightarrow} \stackrel{\sigma}{\Longrightarrow}^{k'} P'$ and $\langle P', Q' \rangle \in \mathcal{R}_{i-k-k'}$.
3. $P \xrightarrow{\sigma} P'$ implies $\exists Q', k{\geq}0.\, k \geq 1{-}i,\ Q \stackrel{\sigma}{\Longrightarrow}^{k} Q'$, and $\langle P', Q' \rangle \in \mathcal{R}_{i-1+k}$.
4. $Q \xrightarrow{\sigma} Q'$ implies $\exists P',\ k{\geq}0.\, k \leq i{+}1,\ P \stackrel{\sigma}{\Longrightarrow}^{k} P'$, and $\langle P', Q' \rangle \in \mathcal{R}_{i+1-k}$.

We write $P \mathrel{\overset{\sqsupset}{\approx}}_i Q$ if $\langle P, Q \rangle \in \mathcal{R}_i$ for some family of weak faster–than relations $(\mathcal{R}_i)_{i \in \mathbb{N}}$, and call $\overset{\sqsupset}{\approx}_0$ the *weak amortized faster–than preorder*.

Relation $\overset{\sqsupset}{\approx}_0$ is indeed a preorder; while reflexivity is obvious, establishing transitivity is simple but nontrivial. The best way of proving transitivity is by showing that $R_k =_{\mathrm{df}} \{ \overset{\sqsupset}{\approx}_i \circ \overset{\sqsupset}{\approx}_j \mid i{+}j = k \}$, for $k \in \mathbb{N}$, is a family of weak faster–than relations. Moreover, one may check that $(\overset{\sqsupset}{\approx}_i)_{i \in \mathbb{N}}$ is the (componentwise) largest family of weak faster–than relations.

Our weakening of the amortized faster–than preorder might appear surprising at first sight, due to the presence of $\stackrel{\sigma}{\Longrightarrow}^{k'}$ trailing weak action transitions on the right–hand side of the definition. As usual for weak bisimilarity, one may have a number of internal transitions before and after a matching action transition, and to get to these trailing internal transitions one may need to pass further clock transitions.

As in the strong case, it is easy to see that $\overset{\sqsupset}{\approx}_0$ is not a precongruence, even not for parallel composition. To identify the largest precongruence contained in $\overset{\sqsupset}{\approx}_0$, one may be tempted to first define a straightforward weak variant of the MT–preorder (with Cond. (3')) and hope that this preorder is included in $\overset{\sqsupset}{\approx}_0$, and is compositional for all operators except summation. This definition would impose the following conditions on the notion of a weak MT–relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$:

1. $P \xrightarrow{\alpha} P'$ implies $\exists Q', k, P'', k'.\, Q \stackrel{\sigma}{\Longrightarrow}^{k} \stackrel{\hat{\alpha}}{\Longrightarrow} \stackrel{\sigma}{\Longrightarrow}^{k'} Q',\ P' \stackrel{\sigma}{\Longrightarrow}^{k+k'} P''$, and $\langle P'', Q' \rangle \in \mathcal{R}$.
2. $Q \xrightarrow{\alpha} Q'$ implies $\exists P'.\, P \stackrel{\hat{\alpha}}{\Longrightarrow} P'$ and $\langle P', Q' \rangle \in \mathcal{R}$.
3. $P \xrightarrow{\sigma} P'$ implies $\exists Q', P'', k.\, Q \stackrel{\sigma}{\Longrightarrow}^{k} Q',\ P' \stackrel{\sigma}{\Longrightarrow}^{k-1} P''$, and $\langle P'', Q' \rangle \in \mathcal{R}$.
4. $Q \xrightarrow{\sigma} Q'$ implies $\exists P'.\, P \stackrel{\sigma}{\Longrightarrow} P'$ and $\langle P', Q' \rangle \in \mathcal{R}$.

Unfortunately, this preorder is not even included in $\overset{\sqsupset}{\approx}_0$, nor is it included in any other desirable weak faster–than preorder. The reason for this is that, e.g., $\tau.(\tau.a.\mathbf{0} + \tau.b.\mathbf{0})$ would be deemed faster than $a.\mathbf{0}$; in particular, the first τ–transition of the faster process to $\tau.a.\mathbf{0} + \tau.b.\mathbf{0}$ can be matched by $a.\mathbf{0} \xrightarrow{\sigma} a.\mathbf{0}$ and choosing $\tau.a.\mathbf{0} + \tau.b.\mathbf{0} \xrightarrow{\tau} a.\mathbf{0} \xrightarrow{\sigma} a.0$. However, $\tau.(\tau.a.\mathbf{0} + \tau.b.\mathbf{0}) \not\mathrel{\overset{\sqsupset}{\approx}}_0 a.\mathbf{0}$, as the transition sequence $\tau.(\tau.a.\mathbf{0} + \tau.b.\mathbf{0}) \xrightarrow{\tau} \tau.a.\mathbf{0} + \tau.b.\mathbf{0} \xrightarrow{\tau} b.\mathbf{0} \xrightarrow{b} \mathbf{0}$ cannot be matched by process $a.\mathbf{0}$. This example indicates that one should demand, in Cond. (1), $P' \xrightarrow{\sigma}^{k+k'} P''$. Similarly, the example $\sigma.(\tau.a.\mathbf{0} + \tau.b.\mathbf{0})$ and $\sigma.\tau.a.\mathbf{0}$ shows that Cond. (3) should be modified to demand $P' \xrightarrow{\sigma}^{k-1} P''$. Furthermore, exploring compositionality for parallel composition suggests also in Cond. (4)

$P \stackrel{\sigma}{\longrightarrow} P'$ (cf. proof of Prop. 18), which implies that we may simply write $Q \stackrel{\sigma}{\longrightarrow} Q'$ and $\langle P', Q' \rangle \in \mathcal{R}$ in Cond. (3) as well. This leads to the following definition of a *weak Moller–Tofts preorder*.

Definition 16 (Weak MT–preorder). A relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is a *weak MT–relation* if, for all $\langle P, Q \rangle \in \mathcal{R}$ and $\alpha \in \mathcal{A}$:

1. $P \stackrel{\alpha}{\longrightarrow} P'$ implies $\exists Q', k, P'', k'.\, Q \stackrel{\sigma}{\Longrightarrow}^{k} \stackrel{\hat{\alpha}}{\Longrightarrow} \stackrel{\sigma}{\Longrightarrow}^{k'} Q',\ P' \stackrel{\sigma}{\longrightarrow}^{k+k'} P''$, and $\langle P'', Q' \rangle \in \mathcal{R}$.
2. $Q \stackrel{\alpha}{\longrightarrow} Q'$ implies $\exists P'.\, P \stackrel{\hat{\alpha}}{\Longrightarrow} P'$ and $\langle P', Q' \rangle \in \mathcal{R}$.
3. $P \stackrel{\sigma}{\longrightarrow} P'$ implies $\exists Q'.\, Q \stackrel{\sigma}{\longrightarrow} Q'$ and $\langle P', Q' \rangle \in \mathcal{R}$.
4. $Q \stackrel{\sigma}{\longrightarrow} Q'$ implies $\exists P'.\, P \stackrel{\sigma}{\longrightarrow} P'$ and $\langle P', Q' \rangle \in \mathcal{R}$.

We write $P \mathrel{\underset{\approx}{\sqsupset}}_{\mathrm{mt}} Q$ if $\langle P, Q \rangle \in \mathcal{R}$ for some weak MT–relation $\mathcal{R}$, and call $\underset{\approx}{\sqsupset}_{\mathrm{mt}}$ the *weak MT–preorder*.

We first show that $\underset{\approx}{\sqsupset}_{\mathrm{mt}}$ is a preorder. While reflexivity is obvious, it is difficult to see whether $\underset{\approx}{\sqsupset}_{\mathrm{mt}}$ is transitive, i.e., whether $\underset{\approx}{\sqsupset}_{\mathrm{mt}} \circ \underset{\approx}{\sqsupset}_{\mathrm{mt}} \subseteq \underset{\approx}{\sqsupset}_{\mathrm{mt}}$ holds. In order to prove transitivity, we first note that $\underset{\approx}{\sqsupset}_{\mathrm{mt}}$ satisfies the property $\underset{\sim}{\sqsupset}_{\mathrm{mt}} \circ \underset{\approx}{\sqsupset}_{\mathrm{mt}} \subseteq \underset{\approx}{\sqsupset}_{\mathrm{mt}}$, to which we refer as *quasi–transitivity*. Next, we establish an important technical lemma for which we need to introduce some notation. For $w, w' \in (\mathcal{A} \cup \{\sigma\})^*$ we write $w \equiv^v w'$ if $w_{\upharpoonright \Lambda \cup \overline{\Lambda}} = w'_{\upharpoonright \Lambda \cup \overline{\Lambda}}$. Intuitively, $w \equiv^v w'$ if the words w, w' are visibly equivalent, i.e., if they are identical up to occurrences of σ and τ. We also let $|w|_\sigma$ denote the number of occurrences of σ in w.

Lemma 17. *Let $Q, Q', R \in \mathcal{P}$ and $w \in (\mathcal{A} \cup \{\sigma\})^*$ with $Q \underset{\approx}{\sqsupset}_{mt} R$ and $Q \stackrel{w}{\longrightarrow} Q'$. Then there exists some Q'', l, R', w'' such that $w \equiv^v w''$, $|w''|_\sigma = |w|_\sigma + l$, $Q' \stackrel{\sigma}{\longrightarrow}^{l} Q''$, $R \stackrel{w''}{\longrightarrow} R'$, and $Q'' \underset{\approx}{\sqsupset}_{mt} R'$.*

Proof. The proof is by induction on the structure of word w. If $w = \epsilon$, then the statement holds trivially. If $w = \sigma v$ for some $v \in (\mathcal{A} \cup \{\sigma\})^*$, then one may easily prove the statement by referring to the induction hypothesis. Hence, we are left with the case $w = \alpha v$. Thus, let process $\hat{Q}$ be such that $Q \stackrel{\alpha}{\longrightarrow} \hat{Q} \stackrel{v}{\longrightarrow} Q'$. By Cond. (1) of Def. 16, there exist processes $R'', \hat{Q}'$, a number $\hat{l}$, and a word w_α with $w_\alpha \equiv^v \alpha$, $|w_\alpha|_\sigma = \hat{l}$, $R \stackrel{w_\alpha}{\longrightarrow} R''$, $\hat{Q} \stackrel{\sigma}{\longrightarrow}^{\hat{l}} \hat{Q}'$, and $\hat{Q}' \underset{\approx}{\sqsupset}_{\mathrm{mt}} R''$. Due to the laziness property, there exists some Q''' with $Q' \stackrel{\sigma}{\longrightarrow}^{\hat{l}} Q'''$. We may now apply Lemma 3(2) to obtain a process $\hat{Q}'''$ satisfying $\hat{Q} \stackrel{\sigma}{\longrightarrow}^{\hat{l}} \hat{Q}' \stackrel{v}{\longrightarrow} \hat{Q}'''$ and $Q''' \underset{\sim}{\sqsupset}_{\mathrm{mt}} \hat{Q}'''$. Applying the induction hypothesis to $\hat{Q}', v, R''$ yields processes $\hat{Q}'', R'$, a number l', and a word v' fulfilling the conditions $v \equiv^v v'$, $|v'|_\sigma = |v|_\sigma + l'$, $\hat{Q}''' \stackrel{\sigma}{\longrightarrow}^{l'} \hat{Q}''$, $R'' \stackrel{v'}{\longrightarrow} R'$, and $\hat{Q}'' \underset{\approx}{\sqsupset}_{\mathrm{mt}} R'$. Since $Q''' \underset{\approx}{\sqsupset}_{\mathrm{mt}} \hat{Q}'''$ and $\hat{Q}''' \stackrel{\sigma}{\longrightarrow}^{l'} \hat{Q}''$ we know by Cond. (4) of Def. 1 of the existence of some process Q'' such that $Q''' \stackrel{\sigma}{\longrightarrow}^{l'} Q''$ and $Q'' \underset{\sim}{\sqsupset}_{\mathrm{mt}} \hat{Q}''$. Thus, $Q'' \underset{\sim}{\sqsupset}_{\mathrm{mt}} \hat{Q}'' \underset{\approx}{\sqsupset}_{\mathrm{mt}} R'$ and, by quasi–transitivity, $Q'' \underset{\approx}{\sqsupset}_{\mathrm{mt}} R'$. By setting $w'' =_{\mathrm{df}} w_\alpha v'$ and $l =_{\mathrm{df}} \hat{l} + l'$ we are done. □

Using this lemma we may now prove the transitivity of the weak MT–preorder.

Proof. (of property $\overset{\sqsupset}{\approx}_{mt} \circ \overset{\sqsupset}{\approx}_{mt} \subseteq \overset{\sqsupset}{\approx}_{mt}$) It is sufficient to show that $\overset{\sqsupset}{\approx}_{\mathrm{mt}} \circ \overset{\sqsupset}{\approx}_{\mathrm{mt}}$ is a weak MT–relation. Let $P \overset{\sqsupset}{\approx}_{\mathrm{mt}} Q \overset{\sqsupset}{\approx}_{\mathrm{mt}} R$ for some processes P, Q, R. We focus only on Cond. (1) of Def. 16, since all other conditions are trivial to establish. Let $P \overset{\alpha}{\longrightarrow} P'$, for which the premise $P \overset{\sqsupset}{\approx}_{\mathrm{mt}} Q$ implies the existence of some Q', k, P'', k' such that $Q \overset{\sigma}{\Longrightarrow}^{k} \overset{\hat{\alpha}}{\Longrightarrow} \overset{\sigma}{\Longrightarrow}^{k'} Q'$, $P' \overset{\sigma}{\longrightarrow}^{k+k'} P''$, and $P'' \overset{\sqsupset}{\approx}_{\mathrm{mt}} Q'$. Further, we apply Lemma 17 to obtain $w'' \in (\mathcal{A} \cup \{\sigma\})^*$, $l \in \mathbb{N}$, $Q'' \in \mathcal{P}$, and $R' \in \mathcal{P}$ such that $w'' \equiv^v \hat{\alpha}$, $|w''|_\sigma = k+k'+l$, $Q' \overset{\sigma}{\longrightarrow}^{l} Q''$, $R \overset{w''}{\longrightarrow} R'$, and $Q'' \overset{\sqsupset}{\approx}_{\mathrm{mt}} R'$. Finally, Cond. (4) of Def. 16 guarantees the existence of some P''' such that $P'' \overset{\sigma}{\longrightarrow}^{l} P'''$ and $P''' \overset{\sqsupset}{\approx}_{\mathrm{mt}} Q''$. Hence, $R \overset{\sigma}{\Longrightarrow}^{l'} \overset{\hat{\alpha}}{\Longrightarrow} \overset{\sigma}{\Longrightarrow}^{l''} R'$ for some $l', l'' \in \mathbb{N}$ with $l'+l'' = k+k'+l$, and $P''' \overset{\sqsupset}{\approx}_{\mathrm{mt}} Q'' \overset{\sqsupset}{\approx}_{\mathrm{mt}} R'$, as desired. □

It is obvious from Defs. 1 and 16 that the MT–preorder $\overset{\sqsupset}{\sim}_{\mathrm{mt}}$ is a weak MT–relation and hence included in the weak MT–preorder $\overset{\sqsupset}{\approx}_{\mathrm{mt}}$. Further, $\overset{\sqsupset}{\approx}_{\mathrm{mt}}$ is included in the weak amortized faster–than preorder $\overset{\sqsupset}{\approx}_{0}$, since one may prove that $\mathcal{R}_i =_{\mathrm{df}} \{\langle P, Q\rangle \mid P \overset{\sigma}{\longrightarrow}^{i} P' \overset{\sqsupset}{\approx}_{\mathrm{mt}} Q\}$ is a family of weak faster–than relations.

Proposition 18. *The weak MT–preorder $\overset{\sqsupset}{\approx}_{mt}$ is compositional for all* TACS$^{\mathrm{LT}}$ *operators except summation.*

Proof. We restrict ourselves to the most interesting case of verifying compositionality of $\overset{\sqsupset}{\approx}_{\mathrm{mt}}$ with respect to parallel composition. To do so, we show that $\mathcal{R} =_{\mathrm{df}} \{\langle P_1|P_2, Q_1|Q_2\rangle \mid P_1 \overset{\sqsupset}{\approx}_{\mathrm{mt}} P_2,\ Q_1 \overset{\sqsupset}{\approx}_{\mathrm{mt}} Q_2\}$ is a weak MT–relation. Let $\langle P_1|P_2, Q_1|Q_2\rangle \in \mathcal{R}$ be arbitrary.

The only difficult part of the proof concerns establishing Cond. (1) of Def. 16 in the case of synchronization. Let $P_1|P_2 \overset{\tau}{\longrightarrow} P_1'|P_2'$ for processes P_1', P_2', due to $P_1 \overset{a}{\longrightarrow} P_1'$ and $P_2 \overset{\overline{a}}{\longrightarrow} P_2'$ for some visible action a. Since $P_1 \overset{\sqsupset}{\approx}_{\mathrm{mt}} Q_1$ we know of the existence of some Q_1', k, P_1'', k' such that $Q_1 \overset{\sigma}{\Longrightarrow}^{k} \overset{a}{\Longrightarrow} \overset{\sigma}{\Longrightarrow}^{k'} Q_1'$, $P_1' \overset{\sigma}{\longrightarrow}^{k+k'} P_1''$, and $P_1'' \overset{\sqsupset}{\approx}_{\mathrm{mt}} Q_1'$. Similarly, since $P_2 \overset{\sqsupset}{\approx}_{\mathrm{mt}} Q_2$ we know of the existence of some Q_2', l, P_2'', l' such that $Q_2 \overset{\sigma}{\Longrightarrow}^{l} \overset{\overline{a}}{\Longrightarrow} \overset{\sigma}{\Longrightarrow}^{l'} Q_2'$, $P_2' \overset{\sigma}{\longrightarrow}^{l+l'} P_2''$, and $P_2'' \overset{\sqsupset}{\approx}_{\mathrm{mt}} Q_2'$. We distinguish the following cases:

- $k = l$: W.l.o.g. we further assume $k' \geq l'$. Due to the laziness property in TACS$^{\mathrm{LT}}$ there exists some Q_2'' with $Q_2' \overset{\sigma}{\longrightarrow}^{k'-l'} Q_2''$ and, because of $P_2'' \overset{\sqsupset}{\approx}_{\mathrm{mt}} Q_2'$, there exists some $\hat{P}_2''$ such that $P_2'' \overset{\sigma}{\longrightarrow}^{k'-l'} \hat{P}_2''$ and $\hat{P}_2'' \overset{\sqsupset}{\approx}_{\mathrm{mt}} Q_2''$. Then, $Q_1|Q_2 \overset{\sigma}{\Longrightarrow}^{k} \overset{\tau}{\Longrightarrow} \overset{\sigma}{\Longrightarrow}^{k'} Q_1'|Q_2''$ and $P_1'|P_2' \overset{\sigma}{\longrightarrow}^{k+k'} P_1''|\hat{P}_2''$ by our operational rules, and $\langle P_1''|\hat{P}_2'', Q_1'|Q_2''\rangle \in \mathcal{R}$ by the definition of $\mathcal{R}$, as desired.
- $k \neq l$: W.l.o.g. we assume $k > l$. We refer to the process between the weak clock transitions and the weak action transition on the path $Q_2 \overset{\sigma}{\Longrightarrow}^{l} \overset{\overline{a}}{\Longrightarrow} \overset{\sigma}{\Longrightarrow}^{l'} Q_2'$ as $\hat{Q}_2$. Because of the laziness property in TACS$^{\mathrm{LT}}$ and since $P_2'' \overset{\sqsupset}{\approx}_{\mathrm{mt}} Q_2'$, there exist processes $\hat{P}_2'', \hat{Q}_2'$ satisfying $P_2'' \overset{\sigma}{\longrightarrow}^{k-l} \hat{P}_2''$, $Q_2' \overset{\sigma}{\longrightarrow}^{k-l} \hat{Q}_2'$ and

$\hat{P}_2'' \underset{\approx}{\sqsupset}_{\mathrm{mt}} \hat{Q}_2'$. (This is the place in this proof we referred to in the last few lines before Definition 16.) We may now apply Lemma 3(2) to obtain some $\hat{Q}_2''$ such that $\hat{Q}_2 \xrightarrow{\sigma}{}^{k-l} \overset{\overline{a}}{\Longrightarrow} \overset{\sigma}{\Longrightarrow}{}^{l'} \hat{Q}_2''$ and $\hat{Q}_2' \underset{\sim}{\sqsupset}_{\mathrm{mt}} \hat{Q}_2''$. Now, $\hat{P}_2'' \underset{\approx}{\sqsupset}_{\mathrm{mt}} \hat{Q}_2' \underset{\sim}{\sqsupset}_{\mathrm{mt}} \hat{Q}_2''$, whence $\hat{P}_2'' \underset{\approx}{\sqsupset}_{\mathrm{mt}} \hat{Q}_2''$ because of $\underset{\sim}{\sqsupset}_{\mathrm{mt}} \subseteq \underset{\approx}{\sqsupset}_{\mathrm{mt}}$ and the transitivity of $\underset{\approx}{\sqsupset}_{\mathrm{mt}}$. Now, we are in the case $k = l$. $\square$

As expected for a CCS–based process calculus, $\underset{\approx}{\sqsupset}_{\mathrm{mt}}$ is not a precongruence for summation, but the summation fix used for other bisimulation–based timed process algebras proves adequate for TACS$^{\mathrm{LT}}$, too.

Definition 19 (Weak MT–precongruence). A relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is a *weak MT–precongruence relation* if, for all $\langle P, Q \rangle \in \mathcal{R}$ and $\alpha \in \mathcal{A}$:

1. $P \xrightarrow{\alpha} P'$ implies $\exists Q', k, P'', k'.\, Q \overset{\sigma}{\Longrightarrow}{}^{k} \overset{\alpha}{\Longrightarrow} \overset{\sigma}{\Longrightarrow}{}^{k'} Q'$, $P' \xrightarrow{\sigma}{}^{k+k'} P''$, and $P'' \underset{\approx}{\sqsupset}_{\mathrm{mt}} Q'$.
2. $Q \xrightarrow{\alpha} Q'$ implies $\exists P'.\, P \overset{\alpha}{\Longrightarrow} P'$ and $P' \underset{\approx}{\sqsupset}_{\mathrm{mt}} Q'$.
3. $P \xrightarrow{\sigma} P'$ implies $\exists Q'.\, Q \xrightarrow{\sigma} Q'$ and $\langle P', Q' \rangle \in \mathcal{R}$.
4. $Q \xrightarrow{\sigma} Q'$ implies $\exists P'.\, P \xrightarrow{\sigma} P'$ and $\langle P', Q' \rangle \in \mathcal{R}$.

We write $P \underset{\approxeq}{\sqsupset}_{\mathrm{mt}} Q$ if $\langle P, Q \rangle \in \mathcal{R}$ for some weak MT–precongruence relation $\mathcal{R}$, and call $\underset{\approxeq}{\sqsupset}_{\mathrm{mt}}$ the *weak MT–precongruence*.

Again, $\underset{\approxeq}{\sqsupset}_{\mathrm{mt}}$ is a preorder and the largest weak MT–precongruence relation. It is worth pointing out that the strong faster–than precongruence $\underset{\sim}{\sqsupset}_{\mathrm{mt}}$ is contained in the weak faster–than precongruence $\underset{\approxeq}{\sqsupset}_{\mathrm{mt}}$, which follows by inspecting the respective definitions. The recursive definition of the weak MT–precongruence employed in (3) and (4) above reflects the fact that clock transitions do not resolve choices.

Theorem 20. *$\underset{\approxeq}{\sqsupset}_{mt}$ is the largest precongruence contained in $\underset{\approx}{\sqsupset}_{mt}$.*

Proof. The proof of compositionality of this preorder regarding the TACS$^{\mathrm{LT}}$ operators is quite standard, except for the parallel composition operator that needs to be treated as for the weak MT–preorder before. Containment is proved by showing that $\underset{\approxeq}{\sqsupset}_{\mathrm{mt}} \cup \underset{\approx}{\sqsupset}_{\mathrm{mt}}$ is a weak MT–relation.

We are left with establishing the "largest" claim. From universal algebra we know that the *largest* precongruence $\underset{\approx}{\sqsupset}{}^{c}_{\mathrm{mt}}$ in $\underset{\approx}{\sqsupset}_{\mathrm{mt}}$ exists and also that $\underset{\approx}{\sqsupset}{}^{c}_{\mathrm{mt}} = \{\langle P, Q \rangle \mid \forall C[x].\, C[P] \underset{\approx}{\sqsupset}_{\mathrm{mt}} C[Q]\}$. Since $\underset{\approxeq}{\sqsupset}_{\mathrm{mt}}$ is a precongruence that is contained in $\underset{\approx}{\sqsupset}_{\mathrm{mt}}$, the inclusion $\underset{\approxeq}{\sqsupset}_{\mathrm{mt}} \subseteq \underset{\approx}{\sqsupset}{}^{c}_{\mathrm{mt}}$ holds. Thus, it remains to show $\underset{\approx}{\sqsupset}{}^{c}_{\mathrm{mt}} \subseteq \underset{\approxeq}{\sqsupset}_{\mathrm{mt}}$. Consider the relation $\underset{\approxeq}{\sqsupset}{}^{aux}_{\mathrm{mt}} =_{\mathrm{df}} \{\langle P, Q \rangle \mid P + c.\mathbf{0} \underset{\approx}{\sqsupset}_{\mathrm{mt}} Q + c.\mathbf{0}\,,\ \text{where } c \notin \mathsf{sort}(P) \cup \mathsf{sort}(Q)\}$. By definition of $\underset{\approxeq}{\sqsupset}{}^{aux}_{\mathrm{mt}}$ we have $\underset{\approx}{\sqsupset}{}^{c}_{\mathrm{mt}} \subseteq \underset{\approxeq}{\sqsupset}{}^{aux}_{\mathrm{mt}}$. We establish the other inclusion $\underset{\approxeq}{\sqsupset}{}^{aux}_{\mathrm{mt}} \subseteq \underset{\approxeq}{\sqsupset}_{\mathrm{mt}}$ by proving that $\underset{\approxeq}{\sqsupset}{}^{aux}_{\mathrm{mt}}$ is a weak MT–precongruence relation. Let $P \underset{\approxeq}{\sqsupset}{}^{aux}_{\mathrm{mt}} Q$, i.e., $P + c.\mathbf{0} \underset{\approx}{\sqsupset}_{\mathrm{mt}} Q + c.\mathbf{0}$, and distinguish the following cases.

– Action transitions: Let $P \stackrel{\alpha}{\longrightarrow} P'$, i.e., $\alpha \neq c$ and $P + c.\mathbf{0} \stackrel{\alpha}{\longrightarrow} P'$ by Rule (Sum1). Since $P \mathrel{\underset{\approx}{\sqsupset}}^{aux}_{\mathrm{mt}} Q$ we conclude the existence of some process R and $k, k' \in \mathbb{N}$ satisfying $Q + c.\mathbf{0} \stackrel{\sigma}{\Longrightarrow}^{k} \stackrel{\hat{\alpha}}{\Longrightarrow} \stackrel{\sigma}{\Longrightarrow}^{k'} R$, $P' \stackrel{\sigma}{\longrightarrow}^{k+k'} P''$ and $P'' \mathrel{\underset{\approx}{\sqsupset}}_{\mathrm{mt}} R$. Since P'' cannot perform a c–transition, $Q + c.\mathbf{0}$ must have performed some action from Q to become R; we conclude $Q \stackrel{\sigma}{\Longrightarrow}^{l} \stackrel{\alpha}{\Longrightarrow} \stackrel{\sigma}{\Longrightarrow}^{l'} R$ with $l+l' = k+k'$. The reverse case, where process Q engages in an action transition, is straightforward, as Conds. (2) of Defs. 16 and 19 coincide with the ones for observation equivalence and observation congruence in CCS [18].
– Clock transitions: Let $P \stackrel{\sigma}{\longrightarrow} P'$. By Rules (tAct) and (tSum), $P + c.\mathbf{0} \stackrel{\sigma}{\longrightarrow} P' + c.\mathbf{0}$ holds. Since $P \mathrel{\underset{\approx}{\sqsupset}}^{aux}_{\mathrm{mt}} Q$ we know of the existence of some process R such that $Q + c.\mathbf{0} \stackrel{\sigma}{\longrightarrow} R$ and $P' + c.\mathbf{0} \mathrel{\underset{\approx}{\sqsupset}}_{\mathrm{mt}} R$. As clock derivatives are unique we have $R \equiv Q' + c.\mathbf{0}$ for some Q' satisfying $Q \stackrel{\sigma}{\longrightarrow} Q'$. Because c is a distinguished action not in the sorts of P' and Q' we may further conclude $P' \mathrel{\underset{\approx}{\sqsupset}}^{aux}_{\mathrm{mt}} Q'$, as desired. The other case, where process Q engages in a clock transition, is analogous.

This shows that $\mathrel{\underset{\approx}{\sqsupset}}^{aux}_{\mathrm{mt}}$ is a weak MT–precongruence relation. Hence, $\mathrel{\underset{\approx}{\sqsupset}}^{aux}_{\mathrm{mt}} \subseteq \mathrel{\underset{\approx}{\sqsupset}}_{\mathrm{mt}}$, as desired. □

It remains an open question whether the weak MT–precongruence is also the largest precongruence contained in the weak amortized faster–than preorder. Our attempts of finding a suitable context for proving this full–abstraction result have been unsuccessful so far. Nevertheless we believe in the validity of such a result and are optimistic to identify a simpler formulation of the weak MT–preorder, referring to fewer internal computation steps, from which the desired context may be derived.

8 Related Work

Although there is a wealth of literature on timed process algebras [6], little work has been done in developing theories for relating processes with respect to speed. The approaches closest to ours are obviously the one by Moller and Tofts regarding processes equipped with lower time bounds [20], and our own one regarding processes equipped with upper time bounds [17]. As these have been referred to and discussed throughout, we refrain from repetitions here.

The probably best–known related work focuses on comparing *process efficiency* rather than process speed. Arun–Kumar and Hennessy [3, 4] have developed a bisimulation–based theory for untimed processes that is based on counting internal actions, which was later carried over to De Nicola and Hennessy's testing framework [12] by Natarajan and Cleaveland [21]. In these theories, runs of parallel processes are seen to be the interleaved runs of their component processes. Consequently, e.g., $(\tau.a.\mathbf{0} \,|\, \tau.\overline{a}.b.\mathbf{0}) \setminus \{a\}$ is as efficient as $\tau.\tau.\tau.b.\mathbf{0}$, whereas, in our setting, $(\sigma.a.\mathbf{0} \,|\, \sigma.\overline{a}.b.\mathbf{0}) \setminus \{a\}$ is strictly faster than $\sigma.\sigma.\tau.b.\mathbf{0}$.

The sparse work on comparing process speeds largely concentrated on worst–case timing behavior on the basis of upper time bounds. Research by Vogler et al. [16, 23] originally employed the concurrency–theoretic framework of Petri nets and testing semantics; it has only recently been carried over to the process algebra PAFAS [11] and is discussed in [17]. Simultaneously, Corradini et al. [10] pursued a different idea for relating processes with respect to speed, which is known as the *ill–timed–but–well–caused* approach [2, 13]. This approach allows system components to attach local time stamps to actions. Since actions may occur as in an untimed process algebra, local time stamps may decrease within a sequence of actions which is exhibited by several processes running in parallel. The presence of these "ill–timed" runs makes it difficult to relate the faster–than preorder of Corradini et al. to the one of Moller and Tofts; a modified approach restricting attention to "well–timed" behaviour might allow some insight.

9 Conclusions and Future Work

In previous work [17], the authors investigated bisimulation–based preorders that relate the speeds of asynchronous processes relative to given upper time bounds, specifying when actions have to be executed at the latest. The present paper considered the case of lower time bounds, specifying when actions may be executed at the earliest, by revisiting the seminal approach of Moller and Tofts [20]. Although Moller and Tofts' work was published more than a decade ago and the first one to introduce a faster–than relation in timed process algebra, it was never followed up in the literature — except for [1] where characteristic formulae are provided. One reason for this might be the absence of strong theoretical results, including the absence of a compositionality result for arbitrary processes, of a full–abstraction result, and of a complete axiomatization for finite processes, as well as the bleak picture drawn in [20] for achieving such results elegantly.

This paper established these nontrivial results by introducing some novel process–algebraic techniques, including a commutation lemma between action and clock transitions. In particular, we proved a full–abstraction theorem with respect to a very intuitive amortized preorder that uses bookkeeping for deciding whether one process is faster than another. In addition, an expansion law was established for finite processes, which paved the way for a sound and complete axiomatization of the Moller–Tofts preorder. This not only testifies to the nature of the MT–preorder but also highlights its importance among the sparse related work in the field. Last, but not least, a variant of the MT–preorder that abstracts from internal, unobservable actions was studied.

Future work should proceed along three directions. First, we wish to complete the theory for our weak MT–precongruence by establishing the conjectured full–abstraction result. Second, the developed preorders should be implemented in a formal verification tool, such as the *Concurrency Workbench NC* [9]. Third, we intend to integrate our theory for lower time bounds with our earlier work on upper time bounds [17], thereby exploring the appropriateness of our faster–than approaches for settings with restricted asynchrony.

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