Is Observational Congruence on μ-Expressions Axiomatisable in Equational Horn Logic?*

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Abstract

It is well known that bisimulation on μ-expressions cannot be finitely axiomatised in equational logic. Complete axiomatisations such as those of Milner and Bloom/Ésik necessarily involve implicational rules. However, both systems rely on features which go beyond pure equational Horn logic: either the rules are impure by involving non-equational side-conditions, or they are schematically infinitary like the congruence rule which is not Horn. It is an open question whether these complications cannot be avoided in the proof-theoretically and computationally clean and powerful setting of second-order equational Horn logic.

This paper presents a positive and a negative result regarding axiomatisability of observational congruence in equational Horn logic. Firstly, we show how Milner's impure rule system can be reworked into a pure Horn axiomatisation that is complete for guarded processes. Secondly, we prove that for unguarded processes, both Milner's and Bloom/Ésik's axiomatisations are incomplete without the congruence rule, and neither system has a complete extension in rank 1 equational axioms. It remains open whether there are higher-rank equational axioms or Horn rules which would render Milner's or Bloom/Ésik's axiomatisations complete for unguarded processes.

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1 Introduction

The existence and nonexistence of equational axiomatisations of behavioural equivalences in process algebra has received significant interest in the literature [2, 8, 23, 24, 26]. Most recent work is concerned with finite processes and equational axiomatisations for a range of operators (such as for priority [1]) and behavioural semantics (such as for simulation equivalence [9]). The focus on finite processes is natural since many behavioural relations cannot be finitely axiomatised in the presence of recursion. This has long been known for regular expressions [11] and was shown to apply to μ-expressions as well [8, 26]. Except for special and not very well understood situations in the language of $*$-expressions [11, 13, 14], purely equational theories appear to be inadequate for recursive processes. Thus, a more powerful setting is needed in order to study the relative proof-theoretic complexities of theories for regular processes.

A suitable and quite natural setting is provided by *(second-order) equational Horn logic* [25]. Indeed, Milner's axiomatisation of strong bisimulation for finite state processes [20] and Bloom/Ésik's abstract generalisation [6, 12] involve *conditional equations*, as does Milner's axiomatisation of *observational congruence* [22] and Glabbeek's axiomatisation of branching bisimulation [16], or the various bisimulation-style equivalences in timed process algebras [3, 4, 5, 10]. Looking at these in detail, however, reveals that they are not strictly Horn theories because they depend on the congruence rule for recursion (cf. rule C4 below) which is not Horn, and they are not pure because rules have guardedness side conditions (cf. rule R2 below).

$$\text{C4}\quad \frac{E = F}{\mu x.\, E = \mu x.\, F} \qquad\qquad \text{R2}\quad \frac{F = E\{F/x\}}{\mu x.\, E = F}\; x \text{ guarded in } E$$

To see that rule C4 is not Horn, consider the soundness of C4 which logically corresponds to the formula $(\forall x.\, E = F) \supset \mu x.\, E = \mu x.\, F$. This formula is not Horn since the precondition of the implication is universally quantified; in [12], this is called an *implication between equations*. The Horn interpretation of C4 would be $\forall x.\ (E = F \supset \mu x.\, E = \mu x.\, F)$ which is unsound. Take for example $E \equiv a.x$ and $F \equiv a.0$. Then, the equation $E\{0/x\} = F\{0/x\}$ is sound, but $\mu x.\, a.x$ is not bisimilar to $\mu x.\, a.0$. In the Horn theory of closed terms, rule C4 is only *admissible* in the sense that, if all closed instantiations of $E = F$ are derivable, then all closed instantiations of $\mu x.\, E = \mu x.\, F$ are derivable, too. But this rule is infinitary and not expressible in Horn form. This leads us to the following – in our opinion – key open problem:

> *Can bisimulation for finite state processes be axiomatised in pure equational Horn logic?*

The answer to this question relates to the issue of guardedness. On the face of it, C4 appears to be necessary to prove equalities between recursive processes. Consider processes $p =_{\mathrm{df}} \mu x.\, (\alpha.x + \beta.x)$ and $q =_{\mathrm{df}} \mu x.\, (\beta.x + \alpha.x)$, where "$\alpha.$" and "$\beta.$" are action prefixes, x is a process variable, "$\mu x.$" the recursion operator and "$+$" nondeterministic choice. The processes p and q are bisimilar, and the equation $p = q$ can be derived by first applying the commutativity law on open terms $\alpha.x+\beta.x = \beta.x+\alpha.x$ and then closing under recursion using C4. Interestingly, for guarded processes, i.e., if both α and β are observable actions, the same is achieved without C4. Using recursive unfolding and commutativity, one derives $p = \alpha.p + \beta.p$ and $q = \alpha.q + \beta.q$, i.e., both p and q provably satisfy the same guarded equation system. From there, by way of rule R2, symmetry and transitivity of equality, one finally gets $p = \mu x.(\alpha.x+\beta.x) = q$.

Due to this issue of unguardedness, the above question is particularly challenging for *observational congruence* [21]. The question's importance lies in the fact that the Horn rule format is crucial for standard automated reasoning based on Prolog-style SLD resolution. Moreover, the question is an interesting one since, as we will show, Bloom and Ésik's axiomatisation which is commonly considered pure Horn is in fact not Horn, and Milner's axiomatisation which is commonly considered impure is in fact pure, i.e., guardedness is equational.

As our first technical result, we provide an axiomatisation of observational congruence for finite state processes which is in pure equational Horn form. This axiomatisation is an adaptation of Milner's proof system and interprets the underlying equality as *partial equivalence* via which we may encode the side condition of rule R2. Our axiom system is sound for all processes and complete for guarded processes. Hence, the question remains whether this axiom system can be extended to handle unguardedness. As our second technical result, we show that no finite rank 1 *equational* extension of Milner's axiom system yields completeness for unguarded processes, not even when including the impure rule R2 or the pure *GA-implication* rule of Bloom/Ésik [26]. To the best of our knowledge, this result is the first negative result on process-algebraic axiomatisations in equational Horn logic to be reported in the literature. It can be generalised to rank 2 and provides a number of technical insights into the proof-theoretic expressiveness of Horn logic for observational congruence. Specifically, we conjecture that unguardedness on μ-expressions cannot be axiomatised in second-order equational Horn logic of any rank. Note that for $*$-expressions this problem does not occur. In [18], Kozen presented a finitary axiomatisation of the Kleene algebra of $*$-expressions involving only pure equational implications. Since $*$-expressions do not have an explicit recursion binder, a congruence rule like C4 is therefore not needed in Kleene algebra.

The proofs of our results can be found in the appendix.

2 The Process Language μBCCSP^2

Variable-binding operators require second-order matching, in order to handle syntactic contexts such as the bodies F of recursive processes $\mu x. F$. This section introduces our process language and makes precise what we understand by a second-order Horn axiomatisation. In particular, the language must be general enough to capture not only the object-level syntax of processes but also the meta-level syntax of schemes and rules needed to formalise logic deduction. Our language μBCCSP^2 is an extension of BCCSP [15] by recursion and schematic variables. It corresponds to the second-order fragment T^2 of [26].

2.1 Second-order Syntax

The second-order language of (schematic, context) μ-*expressions*, or *expressions* for short, is defined by

$$F \quad ::= \quad x(F_1, F_2, \ldots, F_n) \;\mid\; \$\mathsf{k} \;\mid\; 0 \;\mid\; \alpha.F \;\mid\; F_1 + F_2 \;\mid\; \mu x.\, F\,.$$

It includes *variables* x and the usual process-algebraic operators of *prefixing* $\alpha.F$, *summation* $F_1 + F_2$ and *recursion* $\mu x.\, F$. The *prefixes* α range over a denumerable set of *observable actions* $a_0, a_1, a_2, \ldots$ and the distinguished *silent action* τ. The constant 0 represents the *inactive* process. The expressions \$k are *call-back constants*, where $k \geq 1$, which will be used to form contexts.

Every variable x has a *(context) rank* which specifies the number of parameters that x must be instantiated with to form a process. This is done in *(context) applications* of the form $x(F_1, F_2, \ldots, F_n)$, where *rank*$(x) = n$. We assume that there is a countably infinite number of variables at every rank. Rank 0 variables are called *process variables* and all other variables *schematic variables*. For process variables we simply write x instead of $x()$. Recursion is possible over process variables only, i.e., we require *rank*$(x) = 0$ in any expression $\mu x.\, F$. The variable x in $x(F_1, F_2, \ldots, F_n)$ stands for a context with uniquely identified syntactic slots into which the expressions F_i, for $1 \leq i \leq n$, are inserted. These slots are represented by the call-back constants $\$1, \$2, \ldots, \$n$. Formally speaking, call-back constants are nothing but implicitly bound and canonically named process variables. These would be represented as explicit λ-abstractions in higher-order systems like [26]. The result of *instantiating* x by expression F is written $F[F_1, F_2, \ldots, F_n]$ and obtained if each occurrence of \$k in F is substituted by F_k. We say that F has *rank* n if it does not contain call-back constants larger than \$n. Expressions of rank 0 are called *process schemes*, and those of higher rank are called *contexts* or *expressions*. Thus, if F has rank n and all F_i, for $1 \leq i \leq n$, are process schemes, then $F[F_1, F_2, \ldots, F_n]$ is a process scheme.

The recursion operator $\mu x.\, F$ binds all occurrences of process variable x in F. There is no variable binder for schematic variables. The notions of *free* and *bound* occurrences of variables and of *guardedness* of variables are as usual. In particular, a variable x is called guarded in an expression F, if all occurrences of x in F are within the scope of an α-prefix with $\alpha \neq \tau$. An expression F without free variables (of any rank) is *closed*; otherwise it is *open*. Process schemes without schematic variables, i.e., both rank and variable rank are 0, are called *process terms*. Process terms without free process variables are *process constants*, or simply *processes*. We use $E, F, \ldots$ to range over general expressions, $t, u, \ldots$ to range over process terms, and $p, q, \ldots$ for process constants. We let $\equiv$ stand for the syntactic identity on expressions and denote the sub-expression relation by $\trianglelefteq$, i.e., $E \trianglelefteq F$ if either E is a proper sub-expression of F or if $E \equiv F$. Besides the meta-level identity $E \equiv F$ on expressions we consider formal equalities $E = F$ between process schemes, called *equation schemes*. By the *rank* of an equation scheme $E = F$ we understand the maximal variable rank of E and F. As noted above, the rank of a variable specifies the rank of the context expression by which it needs to be instantiated to generate a process scheme.

An *instantiation* σ is a finite partial mapping from variables to expressions which is rank-preserving, i.e., such that for any variable x in the *domain* of σ, expression $\sigma(x)$ is of rank *rank*(x). If E is an expression with free variables X and σ an instantiation with domain X, the *instantiation of* E *by* σ, written $\sigma(E)$, is obtained by recursively replacing each sub-expression $x(F_1, F_2, \ldots, F_n) \trianglelefteq E$ by $\sigma(x)[\sigma(F_1), \sigma(F_2), \ldots, \sigma(F_n)]$. This is a second-order operation which is to be distinguished from the standard first-order *substitution* $E\{F/x\}$ in which variable x is replaced by F in a single recursive pass through E. For instance, if x and y are two variables of rank 0 and 1, respectively, and $E =_{\mathrm{df}} x(y)$, then the substitution σ with $\sigma(x) =_{\mathrm{df}} y + \$1$, $\sigma(y) =_{\mathrm{df}} 0$ yields $\sigma(E) \equiv \sigma(x(y)) \equiv \sigma(x)[\sigma(y)] \equiv (y + \$1)[0] \equiv y + 0$, while substitution $E\{y + \$1/x\}\{0/y\}$ would return $(y + \$1)(0)$ which is not well-formed. Instantiations preserve well-formedness and rank. In particular, if E is a process scheme, then $\sigma(E)$ is again a process scheme.

Preserving well-formedness is not enough for instantiations to be sensible in equational reasoning for recursive processes with variable binding. It must be ensured that in the instantiation $\sigma(E) = \sigma(F)$ of an equation $E = F$ we do not inadvertently capture free process variables inside E or F. An instantiation σ is called *free* for E, if its application $\sigma(E)$ avoids name capture of free process variables, i.e., every occurrence of a free variable in $\sigma(x)$ remains free after instantiation into $\sigma(E)$. We will use symbol θ to range over free instantiations. In practise, there are two options to keep instantiations free. One is to require that θ is *closed*, i.e., for all x in its domain, $\theta(x)$ is closed. The other is to rename bound variables systematically, e.g., by taking expressions up to α-conversion.

2.2 Semantics & Observational Congruence

The semantics of μBCCSP^2 is the transition system induced by process constants as states and where the action–labelled transition relation is inductively defined by the standard operational rules:

$$\frac{-}{\alpha.\,p \xrightarrow{\alpha} p} \qquad \frac{p_1 \xrightarrow{\alpha} q}{p_1 + p_2 \xrightarrow{\alpha} q} \qquad \frac{p_2 \xrightarrow{\alpha} q}{p_1 + p_2 \xrightarrow{\alpha} q} \qquad \frac{t\{\mu x.\,t/x\} \xrightarrow{\alpha} q}{\mu x.\,t \xrightarrow{\alpha} q}$$

Finally, recall the definition of *observational equivalence* and *observational congruence* [21]. As usual, $\overset{\epsilon}{\Longrightarrow}$ stands for $(\xrightarrow{\tau})^*$, $\overset{\alpha}{\Longrightarrow}$ denotes $\overset{\epsilon}{\Longrightarrow} \circ \xrightarrow{\alpha} \circ \overset{\epsilon}{\Longrightarrow}$, and $\hat{\alpha} =_{\mathrm{df}} \alpha$, if $\alpha \neq \tau$, and $\hat{\tau} =_{\mathrm{df}} \epsilon$. A symmetric binary relation $\mathcal{R}$ on process constants is a *weak bisimulation relation* if

$$\forall \langle p, q\rangle \in \mathcal{R}.\ \forall \alpha, p'.\ (p \xrightarrow{\alpha} p' \text{ implies } \exists q'.\, q \overset{\hat{\alpha}}{\Longrightarrow} q' \text{ and } \langle p', q'\rangle \in \mathcal{R}).$$

The largest such relation $\approx$ is an equivalence and referred to as *observational equivalence*. The largest congruence $\cong$ contained in $\approx$, called *observational congruence*, is characterised by the condition that $p \cong q$ iff

$$\forall \alpha, p'.\ (p \xrightarrow{\alpha} p' \text{ implies } \exists q'.\, q \overset{\alpha}{\Longrightarrow} q' \text{ and } p' \approx q'),$$

and symmetrically. The relation $\cong$ is lifted to process schemes E, F by universal abstraction: $E \cong F$, if $\theta(E) \cong \theta(F)$ for all closed instantiations θ.

2.3 Second-order Equational Horn Logic

If E and F are two well-formed process schemes, then formal equations $E = F$ are of second order, also known as *hyper-identities* [8]. This is because of the presence of schematic variables in our setting. A (pure) second-order equational Horn system is a finite set of *Horn rules*, i.e., rules of the form

$$\frac{E_1 = F_1 \quad \cdots \quad E_n = F_n}{E = F} \; ,$$

where the $E_i = F_i$ are referred to as the rule's *premises* and $E = F$ as the rule's *conclusion*. If the rule has no premises, i.e., $n = 0$, then it is called *axiom*. Given a finite set $\mathcal{A}$ of Horn rules, we say that an equation scheme $G = H$ is derivable from $\mathcal{A}$, in symbols $\mathcal{A} \vdash G = H$, if there exists a finite sequence of equation schemes $G_0 = H_0,\ G_1 = H_1,\ \ldots,\ G_n = H_n$ such that (a) $G \equiv G_n$ and $H \equiv H_n$; and (b) every equation $G_i = H_i$ is derived by instantiating some Horn rule

$$\frac{E_1 = F_1 \quad \cdots \quad E_m = F_m}{E = F}$$

from $\mathcal{A}$ by way of a free instantiation θ_i such that (i) $\theta_i(E) \equiv G_i$, $\theta_i(F) \equiv H_i$ and (ii) for all $1 \leq s \leq m$ there exists an index $r < i$ satisfying $\theta_i(E_s) \equiv G_r$ and $\theta_i(F_s) \equiv H_r$. Permitting arbitrary free instantiations yields a rather general notion of deduction for Horn theories. In particular, we can derive equations $\mathcal{A} \vdash t = u$ between open process terms.

Naturally, a theory $\mathcal{A}$ is sound if $\mathcal{A} \vdash G = H$ implies $G \approxeq H$, i.e., $\theta(G) \approxeq \theta(H)$ for all closed instantiations θ. For this to hold true, each Horn rule must be sound in the sense that for all closed instantiations θ, if $\forall i.\, \theta(E_i) \approxeq \theta(F_i)$, then $\theta(E) \approxeq \theta(F)$. This interpretation of soundness, where the universal quantifier over the interpretation of free variables covers the whole rule, is the definitive characteristic of Horn logic. It is important to note that this is something very different from the implication $(\forall i.\, E_i \approxeq F_i) \supset E \approxeq F$, which would be saying that for all closed θ, $\theta(E) \approxeq \theta(F)$ if for all closed θ and i, $\theta(E_i) \approxeq \theta(F_i)$. This is a strictly weaker soundness criterion. The former and stronger Horn-style soundness is the basis for the standard process of Prolog-style SLD resolution, which is known to be complete for Horn theories and ground goals. On open goals $G = H$, the backward proof search generates closed solution instantiations through unification, essentially treating the free variables in the goal as existential or *flexible*. That this works is due to the strong soundness of Horn rules. The difference from the usual first-order setting is that we permit instantiation of schemes by syntactic context functions, which requires *second-order unification*. We refer the reader to [25] for more details on higher-order unification and the proof theory of higher-order Horn logic.

3 A Pure Horn Axiomatisation

This section shows that the side condition "x guarded in E" in Milner's rule R2 can be eliminated in pure equational Horn logic. The key idea is to re-interpret equations so that they only relate *extensional* processes.

The syntactic relation $\unrhd$ of *weak visibility* is the least relation which satisfies the rules $t \unrhd t$ and, if $t \unrhd r$, then $t + u \unrhd r$, $u + t \unrhd r$, $\tau.t \unrhd r$ and $\mu y.\, t \unrhd r\{\mu y.\, t/y\}$. Intuitively, $t \unrhd r$ states that r occurs weakly unguarded in t. Note that $\unrhd$ abstracts from τ-actions unlike the strong form of $\rhd$ in [22, 26]. A process term t is called *extensional* if there is no term $\mu y.\, u$ such that $t \unrhd \mu y.\, u$ and $u \unrhd y$. Hence, an extensional process term is a process term that cannot engage in an initial divergence. For example, process $\mu x.\, (a.x + b.x)$ is extensional whereas $\mu x.\, (a.x + \tau.x)$, $\tau.\mu x.\, (a.x + \tau.x)$ and $\mu x.\, x$ are not. Moreover, every guarded process is extensional, but not vice versa, e.g., $a.\mu x.\, x$ is extensional but not guarded. On the other hand, whenever $\mu x.t$ is extensional, x is guarded in t.

We now provide a sound axiomatisation of $\approxeq$ restricted to extensional processes. This is only a partial equivalence relation, i.e., a relation that is transitive and symmetric but not reflexive. It turns out that with this modification, the side condition of Milner's rule R2 can be expressed purely equationally. In the following, we reconstruct Milner's original axiomatisation of observational congruence as a *pure* equational Horn theory. For notational convenience, we abbreviate the reflexive equation $E = E$ by $E\downarrow$. To begin with, any algebraic axiomatisation depends on reflexivity, symmetry, transitivity and congruence of equality, all of which may be cast into Horn rules:

$$\text{Eq1}\ \frac{-}{0\downarrow} \qquad \text{Eq2}\ \frac{-}{a.x\downarrow} \qquad \text{Eq3}\ \frac{z(\mu x.\, z(x))\downarrow}{\mu x.\, z(x)\downarrow} \qquad \text{Eq4}\ \frac{x = y}{y = x}$$

$$\text{Eq5}\ \frac{x = y \quad y = z}{x = z} \qquad \text{C1}\ \frac{x = y}{\alpha.x = \alpha.y} \qquad \text{C2}\ \frac{x_1 = y_1 \quad x_2 = y_2}{x_1 + x_2 = y_1 + y_2}$$

Here, all x, x_i, y, y_i are process variables, z is a schematic variable of rank 1, α is an arbitrary action and a stands for an action different from τ. (Strictly speaking, for finite axiomatisation, a, α must be read as a special form of action variables.) Eq1–Eq3 are reflexivity rules. Together with Eq4, Eq5, C1 and C2, the above rules yield a weak extension of the standard equational theory for finite processes by recursion. It is weak since it proves $p = p$ for extensional processes only, as shown in Prop. 3.1 below.

The standard equational axioms of commutativity, associativity, idempotence and neutrality, as well as Milner's τ-laws can be phrased as Horn rules, too:

$$\text{S1}\ \frac{x_1 + x_2 = y}{x_2 + x_1 = y} \quad \text{S2}\ \frac{(x_1 + x_2) + x_3 = y}{x_1 + (x_2 + x_3) = y} \quad \text{S3}\ \frac{x = y}{x + y = x} \quad \text{S4}\ \frac{x = y}{x + 0 = y}$$

$$\text{T1}\ \frac{\alpha.x = y}{\alpha.\tau.x = y} \quad \text{T2}\ \frac{\tau.x = y}{x + \tau.x = y} \quad \text{T3}\ \frac{\alpha.(x_1 + \tau.x_2) = y}{\alpha.x_2 + \alpha.(x_1 + \tau.x_2) = y}$$

Finally, consider the following rules for the recursion operator, both of which are variations of Milner's recursion rules [22]:

$$\text{R1}\ \frac{\mu x.\, z(x) = y}{z(\mu x.\, z(x)) = y} \qquad \text{R2}^*\ \frac{x = z(x) \qquad \mu x.\, z(x) = y}{x = y}$$

R1 expresses that $\mu x.\, t$ is a solution of the fixed point equation $x = t$. This is usually represented by an equation scheme $\mu x.\, t = t\{\mu x.\, t/x\}$ for the direct unfolding of recursive processes. Our formulation uses a conditional form which essentially restricts the recursive unfolding to the cases where $\mu x.\, t$ is extensional. The second rule R2* states that extensional equations have unique recursive solutions. More precisely, if $p = t\{p/x\}$ and if the fixed point $\mu x.\, t$ is provably identical to some process q, then p and q are identical. Hence, if $\mu x.\, t$ is extensional, then all solutions of the equation $x = t$ are equal to $\mu x.\, t$. The second premise $\mu x.\, z(x) = y$ of R2* takes the place of the non-equational side condition "x guarded in E" in Milner's R2, in the sense that $\mu x.\, z(x) = y$ can only be derived if z is a guarded context.

Let M* be the system of axioms Eq1–Eq5, C1–C2, S1–S4, T1–T3, R1, R2*. Observe that M* is a pure equational Horn axiomatisation in rank 1.

Proposition 3.1 *A process p is extensional iff M* $\vdash p\downarrow$.*

It is important to note that the statement of Prop. 3.1 is non-monotonic in the number of axioms. Adding axioms to M* may yield provable reflexivities for non-extensional processes, while removing axioms may mean that some extensional processes are not verifiably reflexive any longer.

Theorem 3.2 *M* is sound regarding $\approxeq$ for all processes and complete for guarded processes.*

Proof: The proof is a replay of Milner's proof [22]. For soundness one observes that the side condition of Milner's rule R2 is captured by the equation $\mu x.\, z(x) = y$ in R2*. For if M* $\vdash \mu x.\, E = p$, then by symmetry and transitivity M* $\vdash (\mu x.\, E)\downarrow$, which means that $\mu x.\, E$ is extensional by Prop. 3.1 and thus x is guarded in E. For completeness one observes that, by Prop. 3.1 and since guardedness implies extensionality, M* $\vdash p\downarrow$ is derivable for every guarded process, and also that all rules of Milner can be simulated by the associated rule in M*. ∎

Thm. 3.2 implies that the deductive mechanism of equational (second-order) Horn logic is sufficient to axiomatise recursion on the fragment of (closed) processes. The salient feature of Milner's proof was to show that the infinitary nature of rule C4 can be localised completely in the question of guardedness. Our result shows that the guardedness side condition can be captured equationally in the form of extensionality. Thus, Milner's rank 1 axiomatisation – counter to common belief – is essentially pure Horn. The restriction of completeness to *guarded* processes does not affect expressiveness. Many process algebras (see, e.g., [5]) and tools are based on guarded recursive specifications, and it is well known that every unguarded process is provably equivalent to a guarded one [22]. However, as we shall see next, this latter property seems to depend crucially on the presence of non-Horn rule C4 which is implicit in Milner's original article [22].

4 Can Horn Eliminate Unguardedness?

As seen above, observational congruence of guarded processes in μBCCSP^2 can indeed be formalised in pure equational Horn logic of (variable) rank 1. We now show that for general, unguarded processes, neither Milner's axiomatisation [22] nor Bloom/Ésik's axiomatisation (see Sewell [26]) are complete when leaving out the only non-Horn rule C4. Moreover, both cannot be made complete by adding a finite number of rank 1 equational axioms. We first establish our incompleteness result considering equational axioms, and then lift it to include the standard recursion rules employed by Milner and Bloom/Ésik.

Our plan is to show that certain sound equations cannot be derived from any finite and sound equational axiomatisation of $\approxeq$. These equations involve choices between pairwise distinct actions a_i, for $0 \leq i \leq n-1$, where $n \in \mathbb{N}$. Such a choice can be written in a straightforward way, say as the process $\mathbf{A}_n =_{\mathrm{df}} \tau.\sum_{i=0}^{n-1} a_i.0$, or expressed in a more complex manner through a recursive maze of τ-transitions, each of which postpones the choice without preempting any of the actions a_i. A special class of such terms are constructed from the following family E_k^n of context expressions, indexed by $k \geq 0$ and $n \geq \max(k,1)$:

$$\begin{array}{lcl} E_0^1 & =_{\mathrm{df}} & \$1 \\ E_0^{i+2} & =_{\mathrm{df}} & \$1 + E_0^{i+1}[\$2, \ldots, \$(i{+}2)] \\ E_{j+1}^{i+1} & =_{\mathrm{df}} & \mu x_{i-j}.(\$1 + \tau.\, E_j^{i+1}[\$2, \$3, \ldots, \$(i{+}1), x_{i-j}]), \end{array}$$

where $x_0, x_1, \ldots, x_n$ are pairwise distinct process variables. Each E_k^n is closed and of rank n with k bound variables $x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}$, for instance:

$$\begin{array}{lcl} E_3^3 & \equiv & \mu x_0.\,(\$1 + \tau.\, E_2^3[\$2, \$3, x_0]) \\ & \equiv & \mu x_0.\,(\$1 + \tau.\, \mu x_1.\,(\$2 + \tau.\, E_1^3[\$3, x_0, x_1])) \\ & \equiv & \mu x_0.\,(\$1 + \tau.\, \mu x_1.\,(\$2 + \tau.\, \mu x_2.\,(\$3 + \tau.\, E_0^3[x_0, x_1, x_2]))) \\ & \equiv & \mu x_0.\,(\$1 + \tau.\, \mu x_1.\,(\$2 + \tau.\, \mu x_2.\,(\$3 + \tau.\,(x_0 + (x_1 + x_2))))). \end{array}$$

If $\tilde{a}_0^n$ is a shorthand for the sequence $\tilde{a} =_{\mathrm{df}} a_0.0, a_1.0, \ldots, a_{n-1}.0$, then $E_n^n[\tilde{a}] \approxeq \mathbf{A}_n$. However, as we will see, no finite (rank 1) axiomatisation can derive $E_n^n[\tilde{a}] = \mathbf{A}_n$ for every n. The reason is that the syntactic structure of the E_k^n is judiciously chosen in such a way that they behave atomically under second-order syntactic matching. More specifically, in every solution of an equation $w(\tilde{y}) = E_k^n[\tilde{z}]$ for rank m variable w and process variables $\tilde{y} = y_1, y_2, \ldots, y_m$ and $\tilde{z} = z_0, z_1, \ldots, z_{n-1}$, the context E_k^n must either be contained wholesale in w or in some y_i, rather than be split across w and $\tilde{y}$.

Proposition 4.1 *Let θ be a free instantiation such that $\theta(w)[\theta(\tilde{y})] \equiv E_k^n[\tilde{U}]$ for rank m variable w, process variables $\tilde{y} = y_1, y_2, \ldots, y_m$ and schemes $\tilde{U} = U_0, U_1, \ldots, U_{n-1}$. Then, either $\exists i.\, \theta(y_i) \equiv E_k^n[\tilde{U}]$, or $\exists$ rank m contexts $\tilde{V} = V_0, V_1, \ldots, V_{n-1}$ such that $\theta(w) \equiv E_k^n[\tilde{V}]$ and $V_i[\theta(\tilde{y})] \equiv U_i$, for $0 \leq i \leq n-1$.*

As an example, consider how the expression $E_2^3[a_1.0, a_2.0, x_0] \equiv \mu x_1.\,(a_1.0 + \tau.\, \mu x_2.\,(a_2.0 + \tau.\,(x_0 + (x_1 + x_2))))$ may be matched against the pattern $w(\tilde{y})$ with some instantiation θ. Recalling $E_2^3[a_1.0, a_2.0, x_0] \trianglelefteq E_3^3[\tilde{a}]$, let us further assume that θ is free for $E_3^3[\tilde{a}]$, i.e., $\theta(w)$ must not have variable x_0 free. This means that $E_2^3[a_1.0, a_2.0, x_0]$ cannot be generated from a rank 0 pattern w. In rank 1 there is exactly one nontrivial solution to match the pattern $w(y)$, namely $\theta(w) =_{\mathrm{df}} E_2^3[a_1.0, a_2.0, \$1]$ and $\theta(y) =_{\mathrm{df}} x_0$. Here, 'nontrivial' means that $\theta(w)$ uses at least one call-back but is not identical to it. There are more possibilities for the rank 2 pattern $w(y_1, y_2)$. One of these is $\theta(w) =_{\mathrm{df}} E_2^3[a_1.0, \$1, \$2]$, $\theta(y_1) =_{\mathrm{df}} a_2.0$ and $\theta(y_2) =_{\mathrm{df}} x_0$. Another one is $\theta(w) =_{\mathrm{df}} E_2^3[a_1.\$2, a_2.\,\$2, \$1]$, $\theta(y_1) =_{\mathrm{df}} x_0$ and $\theta(y_2) =_{\mathrm{df}} 0$. The picture is similar for rank 3 pattern $w(y_1, y_2, y_3)$, in the sense that the context E_2^3 is never broken and the call-back arguments $\theta(y_j)$ generate sub-expressions of $a_i.\,0$ with the only constraint that one of $\theta(y_i)$ must be identical to x_0. Now take a look at $E_1^3[a_2.0, x_0, x_1] \trianglelefteq E_3^3[\tilde{a}]$. This time, if θ is to be free for $E_3^3[\tilde{a}]$ again, there is no way in which $E_1^3[a_2.0, x_0, x_1]$ can match the rank 1 pattern $w(y)$. Both variables x_0 and x_1 would have to be introduced by the call-back $\theta(y)$, which is not possible. On the other hand, in rank 2 against pattern $w(y_1, y_2)$ we can find a match by setting $\theta(w) =_{\mathrm{df}} \mu x_2.\,(a_2.0 + \tau.\,(\$1 + (\$2 + x_2))))$, $\theta(y_1) =_{\mathrm{df}} x_0$ and $\theta(y_2) = x_1$.

The importance of Prop. 4.1 is that, if we restrict a scheme G to have at most rank m variables, then in any matching $\theta(G) \equiv E_n^n[\tilde{U}]$ all the contexts E_k^n for $n - k > m$ must either be fully contained in G or fully instantiated via θ from variables in G. For example, if we match $E_3^3[\tilde{a}]$ against a scheme G to find an instantiation θ (free for G) so that $\theta(G) \equiv E_3^3[\tilde{a}]$ then, depending on G either the context E_3^3 is fully contained in G or some context E_k^3, for $0 \leq k \leq 3$, is fully introduced by θ. An example of the first kind would be $G =_{\mathrm{df}} E_3^3[y_0, a_1.\,y_1, a_2.\,y_1]$, $\theta(y_0) =_{\mathrm{df}} a_0.0$ and $\theta(y_1) =_{\mathrm{df}} 0$. Because of what has been discussed above, if E_k^3, for $k = 1, 2, 3$, is to be introduced by θ, we need a variable of rank at least $3 - k$. For instance, $G =_{\mathrm{df}} y$ and $\theta(y) =_{\mathrm{df}} E_3^3[\tilde{a}]$ would introduce E_3^3 wholesale using a rank 0 variable. Further, $G =_{\mathrm{df}} \mu x_0.\,(a_0.0 + \tau.w(x_0))$ and $\theta(w) =_{\mathrm{df}} E_2^3[a_1.0, a_2.0, \$1]$, where w has rank 1, or $G =_{\mathrm{df}} \mu x_0.\,(a_0.0 + \tau.\,\mu x_1.\,w(x_1, x_0))$ with $\theta(w) =_{\mathrm{df}} a_1.0 + \tau.\,E_1^3[a_2.0, \$2, \$1]$ for rank 2, are solutions introducing E_2^3 and E_1^3, respectively, through θ. In other words, under rank restriction, the E_k^n behave atomically with respect to second-order matching. In this paper we shall explore this feature of the indecomposable expressions E_{n-1}^n to prove non-axiomatisability when using only rank 1 schemes. We believe that the families of expressions E_{n-m}^n can be adapted for obtaining non-axiomatisability with respect to maximal rank m, but leave this to future work.

To obtain our negative results we must generalise the processes $E_n^n[\tilde{a}] \approxeq \mathbf{A}_n$ so that they become robust against attempts to transform them under equational reasoning for $\approxeq$. This means that we need to express their essential structural property in slightly more abstract terms. To this end, let Z be a set of variables of rank n and ξ_n^Z the instantiation with domain Z satisfying $\xi_n^Z(z) = E_{n-1}^n$, where $n = \mathit{rank}(z)$. An expression P is called *Z-pure* if P is of rank 0, i.e., a process scheme, and if it does not contain any variables other than those in Z. An action a_i is said to be *i-guarded* in P if each occurrence appears in the i-th argument S_i of some sub-expression $z(S_1, S_2, \ldots, S_n) \trianglelefteq P$.

Definition 4.2 *An expression P is called an n-*pearl *in* shell variables Z *if*

(**P1**) *P is Z-pure (and all $z \in Z$ have rank n).*

(**P2**) *In every sub-expression $z(S_1, S_2, \ldots, S_{n-1}, U) \trianglelefteq P$, for $z \in Z$, U has a free process variable, and all S_i are process constants such that $S_i \approx a_i.0$.*

(**P3**) *There is at least one occurrence of some $z \in Z$ in P, and each action prefix a_i in P, for $i \geq 1$, is i-guarded.*

*An expression S is an n-*shell *if $P \trianglelefteq S$ for some n-pearl P and $\xi_n^Z(S) \approxeq \mathbf{A}_n$. A process p is an n-*noose *if there exists an n-shell S such that $p \equiv \xi_n^z(S)$.*

Since the size information n can be derived from the shell variables Z we will simply talk about *pearls* and *shells* in Z. Note that (P3) implies that a pearl can only contain observable actions a_i for $i \leq n$. Also, $\xi_n^Z(S) \approxeq \mathbf{A}_n$ means that shells S can at most have free variables in Z.

By definition, every n-noose p satisfies $p \approxeq \mathbf{A}_n$. The converse does not hold. Since nooses mix semantic and syntactic properties, they are not in general preserved by observational congruence. For instance, $E_2^2[a_0.0, a_1.0]$ is a 2-noose with shell (and pearl) $S \equiv \mu x_0.\,(a_0.0 + \tau.\,z_1(a_1.0, x_0))$ and shell variables $Z =_{\mathrm{df}} \{z_1\}$, while $\mathbf{A}_2 \equiv \tau.\,(a_0.0 + a_1.0)$ which is observationally congruent to $E_2^2[a_0.0, a_1.0]$ is not a noose. In general, every $E_n^n[\tilde{a}] \approxeq \mathbf{A}_n$ is an n-noose but $\mathbf{A}_n$ is not. Our incompleteness result is based on the observation that, although $E_n^n[\tilde{a}]$ may be transformed under equational reasoning, the property of being an n-noose is hard to break up. Once infected by n-nooses with large n, equational transformations in rank-restricted Horn theories cannot get rid of them. The reason for this is that such proofs always factorise through shells which must be preserved by observational congruence. This is the content of the following propositions.

Proposition 4.3 *Let E and F be two schemes such that $\theta(E) \approxeq \theta(F)$ for all instantiations θ. Then, E is an n-shell iff F is an n-shell.*

Proposition 4.4 *Let E be a scheme of maximal recursion depth rd in which all free variables have maximal rank rk. Suppose rd < 2 and rk $< n-2$, or rd $< n-1$ and rk < 2. Then, every instantiation θ such that $\theta(E)$ is an n-noose can be factorised as $\theta = \xi_n^Z \circ \theta'$ for some instantiation θ' and rank n variables Z in such a way that $\theta'(E)$ is a shell in shell variables Z.*

The proofs of Props. 4.3 and 4.4 involve various auxiliary results about the properties of pearls under semantic equivalence transformations and decomposition by second-order unification. Based on Props. 4.3 and 4.4, the non-axiomatisability result is easily argued using an *intensional* equivalence $p \approxeq_n q$, defined by the condition that $p \approxeq q$ and either both p and q have an n-noose or none of them has. First, one observes that any rank-restricted equational axiomatisation that is sound for $\approxeq$ must also be sound in the intensional sense for large enough n.

Theorem 4.5 *Let $\mathcal{A}$ be a finite second-order equational axiomatisation of maximal variable rank 1 which is sound for $\approxeq$. Then, there exists a natural number m such that for all $n \geq m$, $\mathcal{A} \vdash p = q$ implies $p \approxeq_n q$.*

Proof: Choose m to be larger than the maximal nesting depth of recursions (or the maximal number of free variables) occurring in sub-expressions of any equation of $\mathcal{A}$. Since $\approxeq_n$ is an equivalence, the rules of reflexivity, transitivity and symmetry are sound for $\approxeq_n$. Hence, the statement of Thm. 4.5 follows directly by induction on the length of derivations $\mathcal{A} \vdash p = q$ if we can show that all equational axioms are sound for $\approxeq_n$. To this end, suppose $E = F$ is an axiom of $\mathcal{A}$ and $p \equiv \theta(E)$ and $q \equiv \theta(F)$ for some instantiation θ. If p is an n-noose, then, by Prop. 4.4 and $n \geq m$, there exists an instantiation θ' such that $\theta'(E)$ is an n-shell and $\theta = \xi_n \circ \theta'$. Since $E = F$ is sound for $\approxeq$, the expression $\theta'(F)$ is observationally congruent to the n-shell $\theta'(E)$. This implies $\theta'(F)$ is an n-shell by Prop. 4.3, whence $\theta(F) \equiv \xi_n(\theta'(F)) \equiv q$ is an n-noose. This proves $p \approxeq_n q$. ∎

Now, for any natural number n, we have $E^n_n[\tilde{a}] \approxeq \mathbf{A}_n$. Since $E^n_n[\tilde{a}]$ is an n-noose but $\mathbf{A}_n$ is not, by Thm. 4.5, the sound equation $E^n_n[\tilde{a}] = \mathbf{A}_n$, for large enough n, is not derivable in any finite rank 1 equational axiom system. In other words, any finite system of second-order equational axioms of maximal variable rank 1 which is sound for $\approxeq$ is incomplete. This corollary to Thm. 4.5 is in itself not surprising since it is already known, e.g., from the work of Sewell [26], that pure equational logic is not sufficient to finitely axiomatise (strong) bisimulation on μ-expressions finitely. On the other hand, it is an open problem whether $\approxeq$ can be axiomatised in the more powerful setting of equational Horn logic. As we have seen in Section 3, this is indeed possible for the guarded fragment.

In the following we use Thm. 4.5 to derive two negative results, showing that the two well-known Horn-rules considered by Milner [22] and Bloom/Ésik [6] are incomplete. This is because these rules maintain the intensional equivalence $\approxeq_n$.

4.1 Milner's Rule R2.

Consider Milner's only rule, the folding rule R2, whose soundness depends on a guardedness side condition:

$$\text{R2} \quad \frac{y = z(y)}{y = \mu x.\, z(x)} \quad z \text{ guarded}$$

Theorem 4.6 *There is no finite rank 1 equational extension of Milner's R2 rule (including Milner's axiomatisation without C4) which is sound and complete for $\approxeq$ on unguarded processes.*

Proof: We may assume that any application of R2 instantiates schematic variable z with a nontrivial guarded context, i.e., in which its argument is indeed called behind observable actions. For any instantiation of R2 in which $\theta(z)$ does not invoke its argument (i.e., does not use call-back $1) we have $\theta(z)[p] \equiv \theta(z)$, for any p. One can thus derive $\mu x.\, \theta(z)[x] = \theta(z)$ via the unfolding rule R1, and from there obtain the conclusion $\theta(y) = \mu x.\, \theta(z)[x]$ via standard equational reasoning from the premise $\theta(y) = \theta(z)[\theta(y)]$. In other words, rule R2 becomes redundant for trivial instantiations. Assuming $\theta(z)$ contains $1 we show that R2 can never produce in its conclusion a term that is an n-noose, for any $n \geq 1$.

Suppose R2 is used with instantiation θ such that $\theta(y)$ is an n-noose. By soundness, we would have $\mu x.\, \theta(z)[x] \approxeq \theta(y) \approxeq \theta(z)[\theta(y)]$. However, this cannot be true. The argument $\theta(y)$ of $\theta(z)$ is guarded by an observable action, say b, so that the right-hand side $\theta(z)[\theta(y)]$ has all noose actions a_i from the argument $\theta(y)$ accessible behind b. However, the process $\theta(y)$ on the left-hand side, being an n-noose and thus observationally congruent to $\mathbf{A}_n$, does not perform two actions in sequence. Thus, rule R2 is never applicable when y is instantiated with a process that is an n-noose.

Now suppose that R2 is instantiated so that $\theta(\mu x.\, z(x)) \equiv \mu x.\, \theta(z)[x]$ is an n-noose. Since $\theta(z)$ must have a guarded call-back, this recursion would be able to perform an infinite sequence of actions, which is not possible for nooses. Hence, R2 is sound for $\approxeq_n$, from which Thm. 4.5 obtains Thm. 4.6. ∎

4.2 Bloom and Ésik's "GA-implication" Rule.

Bloom and Ésik [6] presented an implicational axiomatisation using the single rule scheme

$$\text{GA}\ \frac{\mu x.\, w_1(x,x) = \mu x.\, w_2(x,x)}{\mu x.\, w_1(x,x) = \mu x.\, w_2(x, \mu y.\, w_1(x,y))}$$

in two rank 2 variables w_1 and w_2, which is sound and complete for strong bisimulation. As reported in [26], this theory, together with the usual τ-laws [22], is also complete for $\approxeq$. Bloom/Ésik's system is a pure equational theory without side-conditions. However, it still depends on the infinitary congruence rule C4. Again, the reason is that GA preserves large nooses.

Proposition 4.7 *If Bloom/Ésik's rule GA is sound for $\approxeq$, then it is also sound for $\approxeq_n$, for all $n \geq 5$.*

By Thm. 4.5, every sound finite rank 1 equational axiom system is sound for $\approxeq_n$, for some large enough n. Since, by Prop. 4.7, rule GA still preserves $\approxeq_n$, no finite sound equational extension of GA in rank 1 can derive the equality $E^n_n[\tilde{a}] = \mathbf{A}_n$ which is not sound under $\approxeq_n$. Thus, the following theorem holds:

Theorem 4.8 *There is no finite rank 1 equational extension of Bloom/Ésik's rule GA which is sound and complete for $\approxeq$ on unguarded processes.*

5 Discussion, Conclusions & Future Work

We studied the logical basis for equational reasoning about observational congruence on μ-expressions. Pure equational logic is too inexpressive for bisimulation semantics on μ-expressions, while second-order equational Horn logic with its generic rule schemes seems powerful enough to admit finite axiomatisations. Indeed, it is well known that finite axiomatisations for this purpose must employ (second-order) rule schemes [26]. However, this does not mean that those axiomatisations are necessarily pure Horn systems. Specifically, as pointed out here, the congruence rule C4 for the μ-binder is beyond Horn logic. Rule C4 is mostly implicit and taken for granted; however, it breaks the purity of Horn logic and the straightforward applicability of Prolog-style resolution techniques. In particular, it makes the convenient identification of object-level process variables and meta-level schematic variables ("shallow embedding") impossible. Under the traditional point of view, perhaps, the formal complications due to C4 may be considered minimal. Still, the question must be asked whether bisimulation-style equivalences on μ-expressions can in fact be axiomatised in pure Horn logic and thus enjoy the pleasant model-theoretic and proof-theoretic properties of this rather natural logical setting.

This paper undertook some important steps in answering this question. On the positive side, we showed that observational congruence $\approxeq$ can in fact be axiomatised in pure Horn logic for the fragment of *guarded* processes. On the negative side, we proved that Milner's rule R2 and Bloom/Ésik's rule GA, which are known to be complete in the presence of congruence rule C4, cannot be finitely extended by rank 1 equational axioms for *unguarded* processes to yield a complete system for $\approxeq$ without C4. The proof turned out to be highly technical and involved subtle issues in managing second-order unification. However, the effort is well spent since negative results in the more powerful setting of equational Horn logic are potentially more interesting than negative results for pure equational logic. We should mention that we believe that our results do not depend on the cardinality of the action set: provided there exists one action, we can replace the distinct actions a_i by pairwise non-congruent processes.

Our results suggest that pure equational Horn systems are intrinsically limited in dealing with unguarded processes, which applies to both strong bisimulation and observational congruence. For the latter, however, this is more serious since unguardedness across unobservable actions is nontrivial when these are generated dynamically from communication (as in CCS [21]) or hiding (as in CSP [17]). In fact, our work was triggered by failed attempts to obtain a complete axiomatisation of observational congruence for regular processes in the timed process algebras PMC [4] and CSA [10]. In those languages, unguarded processes carry nontrivial semantic behaviour and thus cannot be ignored in the axiomatisation. If it turned out that unguardedness cannot be Horn axiomatised, this would exhibit the intrinsically more difficult proof-theoretic nature of deterministically timed process algebras under observational abstraction.

We believe that the notions of pearls and nooses introduced in this paper can be extended to obtain a negative result for arbitrary rank k equational schemes. As currently defined, our E_k^n contexts would not survive the so-called "diagonal" or "double iteration" identity $\mu x.\, \mu y.\, w(x, y) = \mu x.\, w(x, x)$ (see, e.g., [12]), which has rank 2. Using this scheme together with rank 1 axioms $\mu x.(y + z(x)) = y + \mu x.\, z(x + y)$ and $\mu x.\, \tau.\, z(x) = \tau.\, \mu x.z(\tau.\, x)$ – as well as a finite list of other rank 1 equations for reasoning about processes with a single recursion – would be strong enough to prove $E^n_n[\tilde{a}] = \mathbf{A}_n$. However, we conjecture that by re-defining the E^n_k to rank $n{-}k$ so that $E^{i+1}_{j+1} =_{\mathrm{df}} \mu x_{i-j}.(a_{i-j}.\, x_{i-j} + \tau.\, E^{i+1}_j[\$1, \$2, \ldots, \$(\text{i-j}), x_{i-j}])$, for $j \geq 0$, and E^{i+2}_0 as before in Sec. 4, $E^n_n[\tilde{a}] = \mathbf{A}_n$ cannot be proved using rank 2 equations. We leave the general rank k result and the more difficult case of equational Horn rules other than R2 and GA to future work. Future work shall also investigate whether categorical languages, such as the one used by Bloom and Ésik in [7], can help to simplify our technical framework.

A Proofs for Section 3 and 4

A.1 Proof of Proposition 3.1

The proof of Prop. 3.1 tacitly uses the following simple lemma:

Lemma A.1 *Extensionality can be decided by the following procedure:*

- $\alpha.t$ *is extensional iff* $\alpha \neq \tau$ *or* t *is extensional.*
- $t_1 + t_2$ *is extensional iff both* t_1 *and* t_2 *are extensional.*
- $\mu x.\, t$ *is extensional iff* t *is extensional and* x *occurs only guarded in* t*.*

Furthermore, $t\{\mu x.\, u/x\}$ *is extensional iff* t *is extensional, and* x *is guarded in* t *or* $\mu x.\, u$ *is extensional.*

Proposition 3.1. *A process* p *is extensional iff* $M^* \vdash p \downarrow$*.*

Proof: ($\Rightarrow$) Let t be an extensional process term with at most x as free variable and x is guarded in t. We show by induction on the structure of t that $\vdash t\{\mu x.\, u/x\} \downarrow$ for any term u. From the special case $t \equiv p$, where x is trivially guarded, it follows that $\vdash p \downarrow$ whenever process p is extensional.

Let t be extensional and x guarded in t. Clearly, t cannot be variable x. We only need to consider the cases of nil, prefix, sum and recursion:

- If $t \equiv 0$, we prove $t \downarrow$ by way of rule Eq1.
- Suppose $t \equiv a.t'$. Then, we use rule Eq2 to derive $a.t'\{\mu x.\, u/x\} \downarrow$ as desired.
- If $t \equiv \tau.t'$, then x must be guarded in t' which must be extensional. By induction hypothesis, there is a derivation of $t'\{\mu x.\, u/x\} \downarrow$. Using rule C1 yields $\tau.t'\{\mu x.\, u/x\} \downarrow$.
- If $t \equiv t_1 + t_2$, then x is guarded in t_1 and t_2, which are both extensional. By induction hypothesis, there are derivations for $t_1\{\mu x.\, u/x\} \downarrow$ and $t_2\{\mu x.\, u/x\} \downarrow$. From these we obtain $t_1\{\mu x.\, u/x\} + t_2\{\mu x.\, u/x\} \downarrow$ by rule C2.
- Finally, consider the case $t \equiv \mu y.\, t'$. We must show that $(\mu y.\, t')\{\mu x.\, u/x\} \downarrow$ is derivable, where we may assume that y is not free in u and $x \not\equiv y$. Since t is extensional, y is guarded in t' and t' is extensional. This implies that the recursive unfolding $t'\{\mu y.\, t'/y\}$ is extensional. Also, the assumption that x is guarded in t means that it is guarded in t' and thus also in $t'\{\mu y.\, t'/y\}$. The induction hypothesis now yields a derivation of $t'\{\mu y.\, t'/y\}\{\mu x.\, u/x\} \downarrow$. Since

  $$t'\{\mu y.\, t'/y\}\{\mu x.\, u/x\} \equiv t'\{\mu x.\, u/x\}\{\mu y.\, t'\{\mu x.\, u/x\}/y\}$$

  we can use a suitable instantiation of rule Eq3 to obtain, as desired, a derivation of $\mu y.\, t'\{\mu x.\, u/x\} \downarrow$.

($\Leftarrow$) Again, we prove a slightly more general statement. For all terms t, u and processes q, if $\vdash t\{\mu x.\, u/x\} = q$ or $\vdash q = t\{\mu x.\, u/x\}$, then $t\{\mu x.\, u/x\}$ is extensional. This is the same (cf. Lemma A.1) as proving that t is extensional and either x is guarded in t or $\mu x.\, u$ is extensional. Also note that this induction invariant implies, in particular, that if $\vdash p = q$ then both p and q are extensional.

Suppose that one of (i) $\vdash t\{\mu x.\, u/x\} = q$ or (ii) $\vdash q = t\{\mu x.\, u/x\}$ is true. We show by induction on the structure of these derivations that t is extensional and that further x is guarded in t or else $\mu x.\, u$ is extensional. We consider the last rule that was used to derive (i) or (ii), respectively. Obviously, the induction hypothesis yields the result trivially for (i) and for all rules in which the left-hand term in the equation's conclusion also appears as the left-hand term or right-hand term in one of the premises. This is the case in Eq4, Eq5 and R2. Regarding (ii), the conclusion is trivially obtained from the induction hypothesis in case of rules Eq4, Eq5, S1–S4, T1–T3, R1 and R2. We verify all remaining cases by individual arguments as follows:

- If $t\{\mu x.\, u/x\} \equiv 0$ and both cases (i) and (ii) are by rule Eq1, then we have $q \equiv 0$ and $t \equiv 0$. Hence, t is extensional and x guarded in t.
- If (i) or (ii) are because of rule Eq2, then $t\{\mu x.\, u/x\} \equiv a.p'$ where $t \equiv a.t'$ and $t'\{\mu x.\, u/x\} \equiv p'$. Obviously, t is extensional and x guarded in t.
- Suppose that (i) $t\{\mu x.\, u/x\} \equiv \alpha.p'$, and $\vdash \alpha.p' = q$ is obtained by rule C1 from a strictly smaller derivation $\vdash p' = q'$ such that $q \equiv \alpha.q'$. In this case $t \equiv \alpha.\, t'$ and $p' \equiv t'\{\mu x.\, u/x\}$. By induction hypothesis, t' is extensional and, if x is unguarded in t', then $\mu x.\, u$ is extensional. But then t is extensional, too, and if x unguarded in t, it must be unguarded in t', so that $\mu x.\, u$ extensional, as desired. By symmetry, the same argument applies in case (ii) when $\vdash q = t\{\mu x.\, u/x\}$ is derived using C1.
- Consider that (i) $t\{\mu x.\, u/x\} \equiv p_1 + p_2$ and $\vdash p_1 + p_2 = q$ is obtained by rule C2 from two strictly smaller derivations $\vdash p_1 = q_1$ and $\vdash p_2 = q_2$ such that $q \equiv q_1 + q_2$. Substitution distributes with the summation operator, i.e., $t \equiv t_1 + t_2$ such that $p_1 \equiv t_1\{\mu x.\, u/x\}$ and $p_2 \equiv t_2\{\mu x.\, u/x\}$. The induction hypothesis implies that t_1 and t_2 and thus t are extensional. Further, if x is unguarded in t, it must be unguarded in one of t_1 or t_2, whence $\mu x.\, u$ is extensional by induction hypothesis. This was to be shown. Again, the case (ii) when $\vdash q = t\{\mu x.\, u/x\}$ is obtained by C2 as the last rule is treated symmetrically.
- Suppose $t\{\mu x.\, u/x\} \equiv \mu y.\, v$ for some process variable y and process term v and that $\vdash \mu y.\, v\downarrow$ arises from a smaller derivation $\vdash v\{\mu y.\, v/y\}\downarrow$ by instantiating rule Eq3. Here, cases (i) and (ii) are identical. The induction hypothesis implies that v is extensional and further that $\mu y.\, v$ is extensional if y is unguarded in v. Since $\mu y.\, v$ cannot be extensional without y being guarded, we must have that y is guarded in v. Consequently, $\mu y.\, v$ and thus $t\{\mu x.\, u/x\}$ are extensional, as desired.

- Now consider the case that $t\{\mu x.\, u/x\} \equiv a.p'$ and $\vdash a.p' = a.p'$ arises by rule Eq2. Then, $t \equiv a.t'$ and $p' \equiv t'\{\mu x.\, u/x\}$. Obviously, x is guarded in t and t is extensional, which proves the statement for both cases (i) and (ii) by symmetry.

- Consider that $t\{\mu x.\, u/x\} \equiv t_1\{\mu x.\, u/x\} + t_2\{\mu x.\, u/x\}$ and $\vdash t\{\mu x.\, u/x\} = q$, which is our case (i), arises by rule S1. Then, there must be a smaller derivation $\vdash (t_2 + t_1)\{\mu x.\, u/x\} = q$, which implies that $(t_2 + t_1)\{\mu x.\, u/x\}$ is extensional by induction hypothesis. It is easy to see that this also means that $t\{\mu x.\, u/x\}$ is extensional. As pointed out above, the case of (ii) derived by rule S1, and all the remaining rules, do not need to be considered. From now on we only deal with (i).

- Next, suppose $t \equiv t_1+(t_2+t_3)$ and $\vdash t\{\mu x.\, u/x\} = q$ arises from rule S2. Applying the induction hypothesis yields that $((t_1 + t_2) + t_3)\{\mu x.\, u/x\}$ is extensional. From this it follows that $t\{\mu x.\, u/x\}$ must be extensional.

- If $t \equiv t_1 + t_2$ and $\vdash t\{\mu x.\, u/x\} = q$ is by rule S3, then this must be from a derivation of the equation $t_1\{\mu x.\, u/x\} = t_2\{\mu x.\, u/x\}$. Now we apply the induction hypothesis twice, in both forms (i) and (ii), to conclude that both $t_1\{\mu x.\, u/x\}$ and $t_2\{\mu x.\, u/x\}$ are extensional. Hence, both t_1 and t_2 and thus $t \equiv t_1 + t_2$ are extensional. Further, if x is unguarded in t, it must be unguarded in t_1 or t_2. Either way, the induction hypothesis yields that $\mu x.\, u$ is extensional.

- Consider $t \equiv t_1+0$ and assume that rule S4 is instantiated to give $\vdash t\{\mu x.\, u/x\} = q$ from a smaller derivation of equation $t_1\{\mu x.\, u/x\} = q$. Then, the induction hypothesis obtains that $t_1\{\mu x.\, u/x\}$ is extensional, from which we conclude without difficulty that $t\{\mu x.\, u/x\}$ must be extensional.

- The next rule to look at is T1. Here, for case (i), we would have $t \equiv \alpha.\tau.t'$. The induction hypothesis is applied to a derivation of $(\alpha.t')\{\mu x.\, u/x\} = q$ implying that $\alpha.t'$ is extensional by induction hypothesis. This, however, means that t is extensional, too. Moreover, suppose that x is unguarded in t. Then, $\alpha \equiv \tau$, whence x is unguarded in $\alpha.t'$. By induction hypothesis, then, $\mu x.\, u$ is extensional.

- Rules T2 and T3 are handled by observing that whenever $(\tau.t')\{\mu x.\, u/x\}$ or $(\alpha.(t_1 + \tau.t_2))\{\mu x.\, u/x\}$ are extensional, then so are $(t' + \tau.t')\{\mu x.\, u/x\}$ and $(\alpha.t_2 + \alpha.(t_1 + \tau.t_2))\{\mu x.\, u/x\}$, respectively.

- Suppose that $t\{\mu x.\, u/x\} = q$ has been derived by instantiating rule R1. Hence, there exists a term t' such that $t\{\mu x.\, u/x\} \equiv t'\{\mu x.\, t'/x\}$ and also $\vdash \mu x.\, t' = q$. By induction hypothesis, $\mu x.\, t'$ is extensional. Thus, t' is extensional and x is guarded in t', which finally gives that $t'\{\mu x.\, t'/x\}$ is extensional (all by Lemma A.1), as desired.

∎

A.2 Substitution and Instantiation

To handle our syntax we have two kinds of first-order substitutions, one that replaces variables and another one that replaces call-back constants. For the first kind we employ the usual notation $E\{F/x\}$, whereas for the second kind we introduce new notation as follows. Let F be an expression of rank k, i.e., F may contain the call-back constants 0, \$1, \$2, ..., \$k. These call-back slots can be substituted by a list of expressions $\tilde{E} = E_1, E_2, \ldots, E_r$, and the resulting expression is denoted $F[\tilde{E}]$. This is formally defined as

$$\begin{aligned}
(x(F_1, F_2, \ldots, F_m))[\tilde{E}] &=_{\mathrm{df}} x(F_1[\tilde{E}], F_2[\tilde{E}], \ldots, F_m[\tilde{E}]) \\
\$\mathsf{k}[\tilde{E}] &=_{\mathrm{df}} \begin{cases} E_k & \text{if } 1 \leq k \leq r \\ \$\mathsf{k} & \text{otherwise} \end{cases} \\
(\alpha.\, F)[\tilde{E}] &=_{\mathrm{df}} \alpha.\, (F[\tilde{E}]) \\
(F_1 + F_2)[\tilde{E}] &=_{\mathrm{df}} F_1[\tilde{E}] + F_2[\tilde{E}] \\
(\mu x.\, F)[\tilde{E}] &=_{\mathrm{df}} \mu x.\, (F[\tilde{E}]).
\end{aligned}$$

Thus, $F[\tilde{E}]$ is simply the syntactic substitution $F\{E_1/\$1, E_2/\$2, \ldots, E_r/\$\mathsf{r}\}$. Notice that, in general, r need not be identical to the (minimal) rank of F. If r is larger the unreferenced E_i get dropped; if r is smaller, the resulting expression $F[\tilde{E}]$ will still contain call-back constants.

An *instantiation* with domain X is a finite partial mapping σ which assigns a schematic expression $\sigma(x)$ of rank k to each variable $x \in X$ of rank k. The application of an instantiation (with domain X) to an expression F (with free variables in X) is written $\sigma(F)$ and defined as follows:

$$\begin{aligned}
\sigma(x(\tilde{F})) &=_{\mathrm{df}} \begin{cases} \sigma(x)\,[\sigma(F_1), \sigma(F_2), \ldots, \sigma(F_m)] & \text{if } x \text{ is in the domain of } \sigma \\ x\,[\sigma(F_1), \sigma(F_2), \ldots, \sigma(F_m)] & \text{otherwise} \end{cases} \\
\sigma(\$\mathsf{k}) &=_{\mathrm{df}} \$\mathsf{k} \\
\sigma(\alpha.\, F) &=_{\mathrm{df}} \alpha.\, \sigma(F) \\
\sigma(F_1 + F_2) &=_{\mathrm{df}} \sigma(F_1) + \sigma(F_2) \\
\sigma(\mu x.\, F) &=_{\mathrm{df}} \mu x.\, (\sigma\backslash x)(F),
\end{aligned}$$

where $\sigma\backslash x$ denotes σ with x removed from its domain. We use the same notation to build finite instantiations as substitutions. For example, if G is a rank $k \geq 1$ context and x a rank k variable, then $E\{G/x\}$ denotes the result of recursively replacing all occurrences of $x(\tilde{F})$ in E by $G[\tilde{F}\{G/x\}]$.

The definition of instantiation includes the special case of process variables in the sense that, if x has rank 0, then the instantiation $\sigma(x) \equiv \sigma(x()) \equiv \sigma(x)[]$ would substitute "nothing" for all occurrences of \$0 in $\theta(x)$. We can think of this as turning all occurrences of \$0 into 0. Since this is the only instantiation for the call-back \$0, we identify it with 0 right away. The assumption that instantiations preserve the rank, i.e., that $\sigma(x)$ has the same rank as x, means that whenever E has rank k, then $\sigma(E)$

has rank k, too. In particular, instantiations transform process schemes into process schemes.

Note the difference between substitution and instantiation. If z_1 is a variable of rank 1, then a substitution like $E\{z_2/z_1\}$ can replace z_1 by another rank 1 variable z_2 (without violating the well-formedness of E), although z_2 as an expression has rank 0. On the other hand, in an instantiation $E\{G/z_1\}$, the rank 1 variable z_1 must be substituted by a rank 1 expression G.

An expression is called *k-closed* if it does not have any free variables of rank k, and *closed* if does not contain any free variables whatsoever. The *variable rank* of an expression E is the maximal rank of any free variable in E. If E is of rank 0, i.e., E is a process scheme, then the variable rank of E is simply called the rank of E.

Finally, we need to take some care in the treatment of free and bound (process) variables. For the sound application of rules it is crucial that substitutions or instantiations $E\{F/x\}$ are *free* for the expression E instantiated, i.e., there are no name clashes by which a free variable from F becomes bound by instantiating F for occurrences of x. We shall assume throughout that substitutions and instantiations are free wherever they are applied. This is trivial whenever F is 0-closed. In general, bound variables in E must be renamed if necessary. To make this coherent, expressions must be identified up to α-conversion.

Lemma A.2 *If $U \trianglelefteq E\{F/x\}$, then one of the following holds:*

(i) $U \trianglelefteq E$, or

(ii) $U \trianglelefteq F$ and x is free in E, or

(iii) $U \equiv E'\{F/x\}$ for some $E' \trianglelefteq E$.

Proof: The statement is proved by straightforward induction on E noting that, if x is not free in E, then $U \trianglelefteq E\{F/x\} \equiv E$, whence case (i) of Lem. A.2 holds trivially. ∎

Case (i) in Lem. A.2 covers the situation where U contains a free occurrence of x which is bound in E and distinct from the free occurrence of x substituted in $E\{F/x\}$. For instance, $E \equiv (\mu x.\, \alpha.\, x) + x$ has sub-term $U \equiv \alpha.\, x \trianglelefteq E\{F/x\}$ with, say, $F \equiv 0$, but cases (ii) or (iii) do not apply.

Lemma A.3 *$U \trianglelefteq E[F]$ iff one of the following holds:*

(i) \$1 $\trianglelefteq E$ and $U \trianglelefteq F$, or

(ii) $U \equiv E'[F]$ for some $E' \trianglelefteq E$.

Proof: Straightforward induction on E. ∎

Proposition 4.1. *Let θ be a free instantiation such that $\theta(w)[\theta(\tilde{y})] \equiv E^n_k[\tilde{U}]$ for rank m variable w, process variables $\tilde{y} = y_1, y_2, \ldots, y_m$ and schemes $\tilde{U} = U_0, U_1, \ldots, U_{n-1}$. Then, $\theta(y_i) \equiv E^n_k[\tilde{U}]$ for some i, or there exist rank m contexts $\tilde{V} = V_0, V_1, \ldots, V_{n-1}$ such that $\theta(w) \equiv E^n_k[\tilde{V}]$ and $V_i[\theta(\tilde{y})] \equiv U_i$, for $0 \leq i \leq n-1$.*

Proof: Let $\tilde{p} = p_0, p_1, \ldots, p_{n-1}$ be defined by $p_i =_{\mathrm{df}} \theta(z_i)$ and let $\tilde{G} = G_1, G_2, \ldots, G_m$ be given as $G_i =_{\mathrm{df}} \theta(y_i)$. The identity $\theta(w)[\theta(\tilde{y})] \equiv E^n_k[\theta(\tilde{z})]$ then reads

$$\theta(w)[\tilde{G}] \equiv E^n_k[\tilde{U}]. \tag{1}$$

It will be convenient, for $0 \leq i_1 \leq i_2 \leq n-1$, to denote the sub-sequences $U_{i_1}, U_{i_1+1}, \ldots, U_{i_2}$ and $x_{i_1}, x_{i_1+1}, \ldots, x_{i_2}$ as $\tilde{U}^{i_2}_{i_1}$ and $\tilde{x}^{i_2}_{i_1}$, respectively.

Let us analyse (1) for $\theta(w)$. Obviously, one possibility is that $\theta(w)$ is a call-back \$i and identity (1) is true because of $G_i \equiv E^n_k[\tilde{U}]$. Otherwise, if $\theta(w)$ is non-trivial, it must generate at least the top-level operator of the right-hand side of identity (1). In other words, $\theta(w) \equiv \mu x_{n-k}.\, F_0$ for some scheme F_0 and

$$F_0[\tilde{G}] \;\;\equiv\;\; U_0 + \tau.\, E^n_{k-1}[\tilde{U}^{n-1}_1, x_{n-k}]. \tag{2}$$

Now, F_0 can be a summation or a call-back. In the latter case where $F_0 \equiv$ \$i, say, we would have $G_i \equiv U_0 + \tau.\, E^n_{k-1}[\tilde{U}^{n-1}_1, x_{n-k}]$. This means that G_i would have variable x_{n-k} free, which is bound in $\theta(w) \equiv \mu x_{n-k}.\, F_0$. This contradicts the freeness of G_i. Thus, F_0 must be a summation $F_0 \equiv V_0 + F''_1$ such that $V_0[\tilde{G}] \equiv U_0$ and $F''_1[\tilde{G}] \equiv \tau.\, E^n_{k-1}[\tilde{U}^{n-1}_1, x_{n-k}]$. As before, we argue that F''_1 cannot be a call-back \$i since then $G_i \equiv \tau.\, E^n_{k-1}[\tilde{U}^{n-1}_1, x_{n-k}]$, which would mean that G_i contains x_{n-k} free, thereby contradicting our assumption. Hence, F''_1 is of the form $F''_1 \equiv \tau.\, F'_1$ and $F'_1[\tilde{G}] \equiv E^n_{k-1}[\tilde{U}^{n-1}_1, x_{n-k}]$. Again, F'_1 cannot be a call-back for then $\theta(w)$ would capture a free variable x_{n-k} from $\tilde{G}$. So, further we get $F'_1 \equiv \mu x_{n-k+1}.\, F_1$ for some scheme F_1 and $F_1[\tilde{G}] \equiv U_1 + \tau.\, E^n_{k-2}[\tilde{U}^{n-1}_2, x_{n-k}, x_{n-k+1}]$. We are now in the same situation as before identity (2). By induction we proceed to obtain a sequence of schemes $V_0, V_1, \ldots, V_{k-1}$ such that $V_j[\tilde{G}] \equiv U_j$, for all $0 \leq j \leq k-1$, and F_k such that

$$\begin{aligned}\theta(w) \;\;\equiv\;\; & \mu x_{n-k}.\, V_0 + \tau.\, (\\ & \quad \mu x_{n-k+1}.\, V_1 + \tau.\, (\\ & \quad\quad \mu x_{n-k+2}.\, V_2 + \tau.\, (\\ & \quad\quad\quad \vdots \\ & \quad\quad\quad\quad \mu x_{n-1}.\, (V_{k-1} + \tau.\, F'_k) \cdots)))\end{aligned} \tag{3}$$

and

$$F'_k[\tilde{G}] \;\;\equiv\;\; E^n_0[\tilde{U}^{n-1}_k, x^{n-1}_{n-k}]. \tag{4}$$

From here we continue to analyse F'_k against E^n_0 in a very similar fashion. First we note that $E^n_0[\tilde{U}^{n-1}_k, x^{n-1}_{n-k}]$ is the summation $U_k + E^{n-1}_0[\tilde{U}^{n-1}_{k+1}, x^{n-1}_{n-k}]$ which still has x_{n-k} free.

Therefore, by freeness of θ, the instantiation $F'_k[\tilde{G}]$ cannot generate this expression from one of its call-back arguments G_i. Instead, F'_k must itself produce the summation, i.e., $F'_k \equiv V_k + F'_{k+1}$ such that $V_k[\tilde{G}] \equiv U_k$ as well as $F'_{k+1}[\tilde{G}] \equiv E_0^{n-1}[\tilde{U}_{k+1}^{n-1}, x_{n-k}^{n-1}]$. Obviously, this line of reasoning can be continued from F'_{k+1} to obtain further schemes $V_k, V_{k+1}, \ldots, V_{n-1}$ with $V_j[\tilde{G}] \equiv U_j$, for all $k \leq j \leq n-1$, and F'_n such that

$$F'_k \quad \equiv \quad V_k + (V_{k+1} + (V_{k+2} + \cdots (V_{n-1} + F'_n) \cdots)) \tag{5}$$

as well as

$$F'_n[\tilde{G}] \quad \equiv \quad E_0^{n-k}[x_{n-k}^{n-1}]. \tag{6}$$

At this point we note that the right-hand side $E_0^{n-k}[x_{n-k}^{n-1}]$ of identity (6) is a (right-associated) summation of variables $x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}$ which are all bound by recursions inside $\theta(w)$, as seen in display (3). Thus no call-back G_i in the left-hand side $F'_n[\tilde{G}]$ of identity (6) can contribute to any sub-expression of $E_0^{n-k}[x_{n-k}^{n-1}]$, whence $F'_n \equiv E_0^{n-k}[x_{n-k}^{n-1}]$. This, together with displays (3) and (5) finally means $\theta(w) \equiv E_k^n[\tilde{V}]$ and $\tilde{V}[\tilde{G}] \equiv \tilde{U}$, as stated in Prop 4.1. ∎

A.3 Pearls

Let us use the notation $\mathbf{A}_n$ both for the process constant $\tau. \sum_{i=0}^{n-1} a_i.0$ and for the set of observable actions $\{a_0, a_1, \ldots, a_{n-1}\}$. Further, let $\mathbf{A}_n^+$ denote the set $\mathbf{A}_n \setminus \{a_0\}$. We will generally assume that $n \geq 3$.

For some of the induction proofs it is technically convenient to permit pearls to have free process variables. Therefore, we are going to weaken condition (P1) in the Definition 4.2 of pearls and call P Z-*pure* if it is of rank 0 and Z includes the set of all variables of rank ≥ 1 from P. More precisely, when Z is a set of rank n variables, then P is a pearl in shell variables Z if the following three conditions are met:

(**P1**) P is Z-pure, i.e., P has rank 0 (a process scheme), all observable actions in P are among $\mathbf{A}_n$ and for all variables $z \trianglelefteq P$ we have *rank*$(z) = 0$ or $z \in Z$.

(**P2**) In every sub-expression $z(S_1, S_2, \ldots, S_{n-1}, U) \trianglelefteq P$, for $z \in Z$, the argument U has a free process variable, and all S_i are process constants such that $S_i \approx a_i.0$.

(**P3**) There is at least one occurrence of some $z \in Z$ in P, and each action prefix $a_i \in \mathbf{A}_n^+$ in P, for $i \geq 1$, is i-guarded, i.e., it occurs inside the i-th sub-expression S_i of some context application $z(S_1, S_2, \ldots, S_{n-1}, U) \trianglelefteq P$.

Consequently, the "pearls" from Definition 4.2 must now be referred to as 0-closed pearls. In particular, S is a shell in Z if $\xi_n^Z(S) \approx \mathbf{A}_n$ and $P \trianglelefteq S$ for some 0-*closed* pearl P in Z. In the following, Z is always a set of rank n variables, with the number n being a fixed but arbitrary global constant.

An occurrence $z(\tilde{S}, U) \trianglelefteq E$ of a rank n variable z in an expression E is called a *loop occurrence* if U has a free process variable. We call this a *singleton* loop occurrence if z occurs exactly once inside E. If we only need the assumption that the variable occurs exactly one, then we just say it has a *singleton occurrence*. Note that condition (P2) of pearls states that every $z(\tilde{S}, U) \trianglelefteq P$ in a pearl P is a loop occurrence of z. It will later be convenient to assume that all occurrences of shell variables in P are named apart, i.e., that they are indeed singleton loop occurrences. The other extreme is possible, too. Occasionally, we assume (without loss of generality) that all shell variables are identical.

As is to be expected, much of the developments in this paper rely on the special invariance properties of pearls. First note that both purity and k-closedness are preserved by transitions and sub-expressions, i.e, if P is pure (k-closed, $k \geq 1$) and $P \xrightarrow{\alpha} P'$ or $P' \trianglelefteq P$, then so is P'. For the latter the restriction is important because P' may have process variables free which are bound inside P. Of course, it is for this reason that we have relaxed the notion of purity (P1) in a way that it does not bother about free process variables. It is also obvious that every sub-expression $P' \trianglelefteq P$ of a pearl P not only inherits property (P1) but also (P2). It may, however, fail to satisfy (P3) for two reasons: either it does not contain any further occurrence of a shell variable, which means that P' is a process term (at most free variables of rank 0), or P' contains an action $a_i \in \mathbf{A}_n^+$ which is no longer i-guarded. Here are some important structural invariants for pearls which control those cases:

Lemma A.4 *Let Z be a set of rank n variables.*

(i) If $\alpha. P$ or $\mu x. P$ is a pearl in Z, then so is P.

(ii) If $P \equiv P_1 + P_2$ is a pearl in Z, then at least one of P_i is a pearl in Z, too. Also, if one P_i is not a pearl, then it does not contain any $z \in Z$ and no observable action other than a_0. This means that, if P_i contains some $z \in Z$ or an action prefix $a_i \in \mathbf{A}_n^+$, then it must be a pearl.

(iii) If $P \equiv w(S_1, S_2, \ldots, S_{m-1}, S_m)$ is a pearl in Z, then $w \in Z$, $m = n$ and $S_i \approx a_i.0$ for $a_i \in \mathbf{A}_n$. Further, either S_m contains some $z \in Z$, or S_m is an open process term with at most a_0 as observable action.

Proof: Obvious from the definition. ∎

From Lem. A.4 it follows that a scheme of the form $z(\tilde{S})$ or $a_i. P$ can never be a 0-closed pearl; the former would violate (P2) and the latter (P3). In other words, 0-closed pearls (inside shells) can only be silent prefixes $\tau. P$, summations $P_1 + P_2$ or recursions $\mu x. P$. Note that a pearl might be of the form $a_0. P$, but this is not possible as part of a shell which is observationally congruent to $\mathbf{A}_n$.

Lemma A.5 *Let P be a pearl in shell variables Z. Then, each sub-expression $Q \trianglelefteq P$ which contains a shell variable from Z cannot be i-guarded in P, for $i = 1, \ldots, n-1$, and must be a pearl.*

Proof: Let P, Q, Z be given as in the statement of the Lemma. In particular, suppose $z \trianglelefteq Q$ for some $z \in Z$. As mentioned above, it is easy to see from the definition that every sub-scheme of a pearl satisfies (P1) and (P2). To obtain (P3), too, let $a_i \in \mathbf{A}_n^+$ be any occurrence of an action prefix in Q, i.e., $a_i.\, Q' \trianglelefteq Q$. As P is a pearl, this occurrence must be i-guarded in P, i.e., there is a context application $z(S_1, S_2, \ldots, S_{n-1}, U) \trianglelefteq P$ with $a_i.\, Q' \trianglelefteq S_i$. We claim that the guarding occurrence is actually contained in Q, i.e., that $z(S_1, S_2, \ldots, S_{n-1}, U) \trianglelefteq Q \trianglelefteq P$. Otherwise, if this occurrence of z is in P rather than Q, then the fact that the sub-scheme $a_i.0$ of Q is contained in the immediate sub-scheme S_i of $z(S_1, S_2, \ldots, S_{n-1}, U)$ implies that all of Q is in S_i, i.e., $Q \trianglelefteq S_i$. Yet, this is impossible since $z \trianglelefteq Q$ and $S_i \approx a_i.0$. This shows that Q also fulfils property (P3) of pearls. It is obvious that a pearl Q inside another pearl P cannot be i-guarded ($1 \leq i \leq n-1$), since it would then have to be weakly bisimilar to $a_i.0$ which is not the case. ∎

Finally, the following lemma relates n-nooses and pearls:

Lemma A.6 *Let p be a process constant such that $p \approx \mathbf{A}_n$. Then, p is an n-noose iff there exists a 0-closed pearl in some shell variables Z such that $\xi_n^Z(P) \trianglelefteq p$.*

Proof: ($\Leftarrow$) We must show that there exists an n-shell S such that $p \equiv \xi_n^Z(S)$ for some shell variables Z. To this end we observe that $\xi_n^Z(P) \trianglelefteq p$ implies $p \equiv t\{\xi_n^Z(P)/y\} \equiv \xi_n^Z(t\{P/y\})$ for some process term t in free variable y. By definition and the $p \approx \mathbf{A}_n$ assumption, $t\{P/y\}$ is an n-shell.

($\Rightarrow$) If p is an n-noose, then, by definition of nooses, there is an n-shell S with shell variables Z such that $\xi_n^Z(S) \equiv p$. By definition of shells, S contains a 0-closed pearl P, i.e., $S \equiv T\{P/v\}$ where $v \trianglelefteq T$ is a free process variable. Thus, $\xi_n^Z(S) \equiv t\{\xi_n^Z(P)/v\}$, where $t =_{\mathrm{df}} \xi_n^Z(T)$ has no free variables other than v. From this we get $\xi_n^Z(P) \trianglelefteq p$, as desired. ∎

A.4 Symbolic Semantics for Process Schemes

It is our aim to show that the deduction mechanics of second-order equational Horn logic necessarily preserves some abstract structural invariants in the syntactic shape of the process schemes it manipulates. For the present work, these structural properties are cast into the notion of n-shell. The non-axiomatisation argument is based on the observation that any derivation $\mathcal{A} \vdash p = q$ of an equality in which one of the two processes, say p is an instantiation $p \equiv \theta_p(S_p)$ of an n-shell S, the other process q, too, must arise from an n-shell S_p, so that $q \equiv \theta_q(S_q)$. Indeed, we show by induction on derivation that the equality $\mathcal{A} \vdash p = q$ is actually an identity between the underlying shells, so that we really have a proof of an equivalence $\mathcal{A} \vdash S_p = S_q$ at the schematic level. As in the work of Sewell [26], it is therefore natural to lift the operational semantics to the schematic level and use a more abstract symbolic notion of bisimulation. This avoids having to handle the semantics of schematic equations like $S_p = S_q$ using explicit universal quantification over all closed instantiations, i.e., by unfolding to $\forall \theta.\, \theta(S_p) \approxeq \theta(S_q)$.

In this vein we now introduce a symbolic transition semantics for process schemes. We treat occurrences $x(E_1, E_2, \ldots, E_m)$ of variables x as a new form of symbolic action prefixes which provide observable and indexed access to their arguments E_i. Besides the $a_i \in \mathbf{A}_n$ and the silent action τ, our set of actions now includes also symbolic actions of the shape (x, i), where x is a variable and $0 \leq i \leq \mathit{rank}(x)$. A symbolic prefix $x(E_1, E_2, \ldots, E_m)$ then generates a transition with label (x, i) to E_i. As a special case, we have action $(x, 0)$ for a process variable x. Remembering that x is a shorthand for $x()$, process variable x performs a transition labelled $(x, 0)$ to $()$, which means we need to consider $()$ as a special process constant. We thus get the following obvious operational rules:

$$\frac{—}{\alpha.\, E \stackrel{\alpha}{\longrightarrow} E} \qquad \frac{E_1 \stackrel{\alpha}{\longrightarrow} F}{E_1 + E_2 \stackrel{\alpha}{\longrightarrow} F} \qquad \frac{E_2 \stackrel{\alpha}{\longrightarrow} F}{E_1 + E_2 \stackrel{\alpha}{\longrightarrow} F}$$

$$\frac{E\{\mu x.\, E/x\} \stackrel{\alpha}{\longrightarrow} F}{\mu x.\, E \stackrel{\alpha}{\longrightarrow} F}$$

$$\frac{—}{x(E_1, E_2, \ldots, E_m) \stackrel{(x,i)}{\longrightarrow} E_i}\; m \geq i \geq 1 \qquad \frac{—}{x \stackrel{(x,0)}{\longrightarrow} ()}$$

For convenience, we will use the same symbol $\longrightarrow$ for the transitive and reflexive closure of the basic one-step transition. So, if $t = \alpha_1 \alpha_2 \cdots \alpha_k$ is an action sequence, $E \stackrel{t}{\longrightarrow} F$ means that there exist processes $E_0, E_1, \ldots, E_k$ such that $E_{i-1} \stackrel{\alpha_i}{\longrightarrow} E_i$ for all $1 \leq i \leq k$, where $E_0 =_{\mathrm{df}} E$ and $E_k \equiv F$. The reflexive case $E \equiv F$ is written $E \stackrel{\epsilon}{\longrightarrow} F$. Similarly, we define the weak transition relation $\Longrightarrow$ which abstracts from all silent τ steps. Whenever the action sequence t becomes unwieldy to put above the transition arrow, we also write $t : E \longrightarrow F$ and $t : E \Longrightarrow F$ instead of $E \stackrel{t}{\longrightarrow} F$ and $E \stackrel{t}{\Longrightarrow} F$, respectively.

The notions of observational equivalence $\approx$ and observational congruence $\approxeq$ generalise to schematic processes based on the above transition semantics in the natural way. Our treatment of variables as actions provides an alternative to the visibility relation $\rhd$ used elsewhere [22, 26]. For more information specifically on higher-order symbolic simulation, see [26].

A.5 Proof of Proposition 4.3

In this section, all expressions are process schemes, i.e., have 0 rank. The first lemma is analogous to Lem. A.2. It analyses the operational semantics of substitution. It uses the relation $\rhd$ of *strong visibility* [22, 26]. Specifically, $E \rhd x$ if x occurs in top-level position, unguarded by any action or variable of rank ≥ 1. It is the strong form of $\unrhd$ (see the beginning of Sec. 3).

Lemma A.7 *$E\{P/x\} \stackrel{\alpha}{\longrightarrow} F$ iff one of the following holds:*

(i) $P \stackrel{\alpha}{\longrightarrow} F$ and $E \rhd x$, or

(ii) $E \stackrel{\alpha}{\longrightarrow} F'$ *for some* F' *and* $F \equiv F'\{P/x\}$.

Further, in case $P \equiv \mu x. E$*, the statement can be strengthened by dropping the first option above.*

Proof: By induction on the structure of derivations for the transition relation $\stackrel{\alpha}{\longrightarrow}$, see e.g. [26][Prop. 3, Lem. 4, 5, 35]. ∎

Lemma A.8 *A process variable* x *occurs free in* E *iff there exists an action sequence* t *and expression* F *such that* $E \stackrel{t}{\longrightarrow} F$ *and* $F \rhd x$.

Proof: Direction ($\Rightarrow$) is straightforward by induction on E, using Lem. A.7 and the fact that, if $F \rhd x$ and $y \neq x$, then $F\{E/y\} \rhd x$.

The reverse direction ($\Leftarrow$) is obtained by structural induction on the derivation $E \stackrel{t}{\longrightarrow} F$. Let us look at the interesting case which is recursion, i.e., $E \equiv \mu y. E'$ and $E \stackrel{t}{\longrightarrow} F$ such that $F \rhd x$. Then, $E'\{E/y\} \stackrel{t}{\longrightarrow} F$ by the operational rule for recursion, and the induction hypothesis tells us that x is free in $E'\{E/y\}$. We must have $x \neq y$ because otherwise, on the one hand, the fact that x is free in $E'\{E/y\} \equiv E'\{E/x\}$ would imply that x is free in E. On the other hand, $E \equiv \mu y. E' \equiv \mu x. E'$ binds x by recursion, i.e., it does not have x free, which is a contradiction. Therefore, $x \neq y$ as claimed. Next consider that x could be free in $E'\{E/y\}$ for two reasons: either y occurs free in E' and x is free in E, in which case we are done, or x is free in E' which means it is also free in $E \equiv \mu y. E'$ as $x \neq y$. ∎

Lemma A.9 *Let* $P \trianglelefteq E$ *be a sub-expression of* E *such that* $P \stackrel{\alpha}{\longrightarrow} Q$*. Then,* $E \stackrel{t\alpha}{\longrightarrow} Q'$ *for some transition sequence* t *and process substitution instance* Q' *of* Q.

Proof: By induction on the structure of E we prove that $E \stackrel{t\alpha}{\longrightarrow} Q'$ for some action sequence t and substitution instance Q' of Q:

- If $E \equiv P$, the result follows trivially by assumption with transition sequence $t = \epsilon$ and $Q' \equiv Q$.
- If $E \equiv \beta. G$ and $P \trianglelefteq G$, then $G \stackrel{t\alpha}{\longrightarrow} Q'$ for some Q' and t by induction hypothesis. From this, obviously, we get $E \equiv \beta. G \stackrel{\beta}{\longrightarrow} G \stackrel{t\alpha}{\longrightarrow} Q'$ and thus $E \stackrel{\beta t\alpha}{\longrightarrow} Q'$ overall.
- If $E \equiv w(G_1, G_2, \ldots, G_k)$, where $k = \mathit{rank}(w)$ and $P \trianglelefteq G_i$, then $G_i \stackrel{t\alpha}{\longrightarrow} Q'$ for some Q' and t by induction hypothesis. From this, the transition sequence $(w, i)t\alpha : E \longrightarrow Q'$ emerges.
- Suppose $E \equiv G_1 + G_2$ and $P \trianglelefteq G_1$ or $P \trianglelefteq G_2$. In the former case, the induction hypothesis yields $G_1 \stackrel{t\alpha}{\longrightarrow} Q'$ for some Q' and t. Since the transition sequence $t\alpha$ is non-empty we may conclude $E \equiv G_1 + G_2 \stackrel{t\alpha}{\longrightarrow} Q'$, as desired. In the latter case, where $P \trianglelefteq G_2$, we argue analogously for G_2.

- If $E \equiv \mu x. G$ and $P \trianglelefteq G$, then $G \stackrel{t\alpha}{\longrightarrow} Q'$ by induction hypothesis, from which Lemma A.7 (case (ii) in direction $\Leftarrow$) for transitions permits us to infer

  $$G\{\mu x. G/x\} \stackrel{t\alpha}{\longrightarrow} Q'\{\mu x. G/x\}.$$

  But this means that $E \equiv \mu x. G \stackrel{t\alpha}{\longrightarrow} Q'\{\mu x. G/x\}$ which is what we need since, if Q' is a substitution instance of Q, then $Q'\{\mu x. G/x\}$ is one, too.

∎

We now show that 0-closed pearls cannot miraculously be generated by action transitions:

Lemma A.10 *Let $E \stackrel{\alpha}{\longrightarrow} F$ such that E is 0-closed and F contains a 0-closed pearl. Then, E contains a 0-closed pearl, too.*

Proof: We prove the statement by induction on the structure of the derivation $E \stackrel{\alpha}{\longrightarrow} F$. Note that E cannot be a process variable x and $\alpha = (x, 0)$, for then $F \equiv ()$ which is not possible by assumption.

- If $E \equiv \alpha. F$, then we are done since every sub-expression of F is a sub-expression of E.

- If $E \equiv w(G_1, G_2, \ldots, G_k)$, where $k = \mathit{rank}(w)$, then $\alpha = (w, i)$ and $F \equiv G_i$. Here, too we are done immediately as E inherits the 0-closed pearl from its direct sub-expression G_i.

- If $E \equiv G_1 + G_2$ and $G_1 \stackrel{\alpha}{\longrightarrow} F$, then the induction hypothesis yields a 0-closed pearl $P \trianglelefteq G_1$ which is also a sub-expression of E. The same reasoning applies if $G_2 \stackrel{\alpha}{\longrightarrow} F$.

- Let $E \equiv \mu x. G$ and $G\{E/x\} \stackrel{\alpha}{\longrightarrow} F$. Using the induction hypothesis and observing that $G\{E/x\}$ is again 0-closed, we get a 0-closed pearl $P \trianglelefteq G\{E/x\}$. By Lem. A.2 we have one of the following: (i) $P \trianglelefteq G$, (ii) $P \trianglelefteq E$ and x is free in G or (iii) there is $G' \trianglelefteq G$ with $P \equiv G'\{E/x\}$. In the second case we are done immediately. In the third case, if x is not free in G', the result follows directly, too, since then $P \equiv G'\{E/x\} \equiv G' \trianglelefteq G \trianglelefteq E$. Similarly, in the first case. So we may assume that x is free in G'. Moreover, we conclude that $E \equiv \mu x. G$ must contain at least one occurrence of z. Otherwise, if there is none, G would not have any z and thus neither the substitution $G\{E/x\}$ nor its sub-expression $P \trianglelefteq G\{E/x\}$. But this contradicts the fact that P is a 0-closed pearl. Hence, we find x is free in G' and $z \trianglelefteq E$. Since E is 0-closed by assumption we may safely invoke Lemma A.5 and conclude that E must be a 0-closed pearl, so that $E \trianglelefteq E$ trivially completes the proof.

∎

The next lemma studies the invariance of (P2) under transitions:

Lemma A.11 *Let P be a pearl in shell variables Z and $P \xrightarrow{\alpha} Q$. Consider any $z(\tilde{S}, R) \trianglelefteq Q$, for shell variable $z \in Z$ and schemes $\tilde{S} = S_1, S_2, \ldots, S_{n-1}$ and R. Then,*

(i) $S_i \approx a_i.0$ for all $1 \leq i \leq n-1$ and

(ii) R is not 0-closed or not n-closed, i.e., R has a free process variable or $z \trianglelefteq R$ for some $z \in Z$.

Proof: Let P, Q, R, z, Z and $\tilde{S} = S_1, S_2, \ldots, S_{n-1}$ be given such that $P \xrightarrow{\alpha} Q$ and $z(\tilde{S}, R) \trianglelefteq Q$. Observe that $z \trianglelefteq Q$ means that P cannot be a process variable. We argue for (i) and (ii) by induction on the structure of P, considering merely the remaining cases of prefixes, context application, summation and recursion:

- If $P \equiv \alpha.Q$, then $z(\tilde{S}, R) \trianglelefteq Q \trianglelefteq \alpha.Q \equiv P$, and since P is a pearl satisfying (P2) both statements (i) and (ii) follow.
- If $P \equiv w(P_1, P_2, \ldots, P_n)$ and $\alpha = (w, i)$ with $P_i \equiv Q$, we argue in exactly the same way: the sub-scheme $z(\tilde{S}, R)$ of Q is trivially a sub-scheme of P and thus (P2) applies.
- For summation $P \equiv P_1 + P_2$ and $P_1 \xrightarrow{\alpha} Q$ or $P_2 \xrightarrow{\alpha} Q$ we simply invoke the induction hypothesis, noting that $z \trianglelefteq Q$ implies $z \trianglelefteq P_i$ and thus, by Lem. A.4, P_i is a pearl.
- Let $P \equiv \mu x.E$ and $E\{P/x\} \xrightarrow{\alpha} Q$. From Lem. A.4(i) we conclude without difficulties that E must be a pearl in shell variables Z. By Lemma A.7, there is F such that $Q \equiv F\{P/x\}$ and $E \xrightarrow{\alpha} F$ (note that, due to the special structure of $P \equiv \mu x.E$, case (i) of Lemma A.7 is excluded). Now apply Lem. A.2 to $z(\tilde{S}, R) \trianglelefteq Q \equiv F\{P/x\}$ which yields three cases:
  (i) If $z(\tilde{S}, R) \trianglelefteq F$, we can use the induction hypothesis for $E \xrightarrow{\alpha} F$ directly to prove the desired statement,
  (ii) If $z(\tilde{S}, R) \trianglelefteq P$, then the result is obtained by exploiting that P is a pearl.
  (iii) The final case is where we have a sub-scheme $F' \trianglelefteq F$ and $z(\tilde{S}, R) \equiv F'\{P/x\}$. Observe that F' cannot be variable x since then $P \equiv z(\tilde{S}, R)$, contradicting the assumption $P \equiv \mu x.E$. Thus, there are $\tilde{S}' = S'_1, S'_2, \ldots, S'_{n-1}, R'$ such that $F' \equiv z(\tilde{S}', R')$, $S_i \equiv S'_i\{P/x\}$ and $R \equiv R'\{P/x\}$. Since $z(\tilde{S}', R') \equiv F' \trianglelefteq F$ we can invoke the induction hypothesis and conclude that (i) R' has a free process variable or contains some shell variable from Z and (ii) $S'_i \approx a_i.0$. The latter implies that S'_i cannot have x free, whence $S_i \equiv S'_i\{P/x\} \equiv S'_i \approx a_i.0$ as desired. Similarly, the former can only be true if R has a free process variable or a shell variable. Note that, if the free process variable of R' happens to be x and any free occurrence of x in R gets eliminated in $R \equiv R'\{P/x\}$, then at least R inherits all shell variables from $P \trianglelefteq R$.

∎

Pearls are not invariant under transitions. Naturally, if $P \xrightarrow{a_i} Q$ for $a_i \in \mathbf{A}_n^+$, then $Q \approx 0$ by construction of pearls. Also, the unrolling of a recursion

$$\mu x_0.(a_0.0 + \tau.\, z(\tilde{a}, x_0)) \xrightarrow{\tau} z(\tilde{a}, \mu x_0.(a_0.0 + \tau.\, z(\tilde{a}, x_0)))$$

destroys property (P2). However, there is a weak form of preservation which is expressed in the following lemma:

Lemma A.12 *Let P be a 0-closed pearl in shell variables Z and $P \xrightarrow{\alpha} Q$ such that $z \trianglelefteq Q$ for some $z \in Q$. Then, Q is a 0-closed pearl, or there is a sub-expression $z(\tilde{S}, R) \trianglelefteq Q$ in which R is a 0-closed pearl.*

Proof: We prove the lemma by induction on the syntactic structure of P when fixing shell variables Z, i.e., all pearls will be pearls in Z. To make the inductive argument go smoothly we prove the following slightly modified statement:

> *Let P be a pearl such that $P \xrightarrow{\alpha} Q$ and $z \trianglelefteq Q$ for some $z \in Z$ and 0-closed Q. Then, Q is a pearl, or there is $z(\tilde{S}, R) \trianglelefteq Q$ in which R is a 0-closed pearl.*

Clearly, the statement of Lemma A.12 is a special case of this because 0-closedness is preserved by transitions. Observe that P cannot be a process variable. Using these restrictions, the proof proceeds by induction as follows:

- If $P \equiv \alpha.\, Q$ then the fact that P is a pearl implies that Q is a pearl (Lem. A.4(i)).
- Let $P \equiv w(S_1, S_2, \ldots, S_{n-1}, R)$ for $w \in Z$. The option that $\alpha = (w, i)$, for $i = 1, \ldots, n-1$, would mean $S_i \equiv Q$. But since P is a pearl $S_i \approx a_i.0$ which contradicts $z \trianglelefteq Q$. Hence, we must have $\alpha = (w, n)$ and $R \equiv Q$. The fact that $z \trianglelefteq Q$ then implies that Q is a pearl (Lem. A.4(iii)).
- If $P \equiv P_1 + P_2$ and $P_1 \xrightarrow{\alpha} Q$. By assumption $z \trianglelefteq Q$, which means that $z \trianglelefteq P_1$, as well, because transitions do not introduce new free variables. But then Lem. A.4(ii) says that P_1 is a pearl and thus the desired result about Q is obtained directly from the induction hypothesis. The same holds if $P_2 \xrightarrow{\alpha} Q$.
- Suppose $P \equiv \mu x.\, E$ and $E\{P/x\} \xrightarrow{\alpha} Q$. From Lem. A.4(i) we conclude as before that E is a pearl and further, by Lemma A.7, that there is F with $Q \equiv F\{P/x\}$ and $E \xrightarrow{\alpha} F$.

  If x is not free in F, then $Q \equiv F$ and thus $E \xrightarrow{\alpha} Q$. Then, the induction hypothesis finishes the argument, i.e., we find that Q is a pearl or it contains $z(\tilde{S}, R) \trianglelefteq Q$ such that R is a 0-closed pearl.

  From now on assume that x occurs free in F, i.e., $P \trianglelefteq Q$. Note that $Q \equiv F\{P/x\}$ since a transition successor of pearl P satisfies (P1). The same is true for all sub-schemes of Q. We claim that Q also satisfies condition (P3) of pearls.

To start off, $z \trianglelefteq Q$ by assumption. Also, we cannot have $\alpha = (w, i)$ for $w \in Z$ and $1 \leq i \leq n-1$ since then, by the property of the pearl P, we would have $Q \approx a_i.0$ which contradicts $z \trianglelefteq Q$. This leaves $\alpha = (z, n)$ as the only possibility for a context action. But then, every prefix $a_i \in \mathbf{A}_n^+$ must still be i-guarded in Q. Thus, Q indeed fulfils both conditions (P1) and (P3) of a pearl. Regarding (P2), consider any $z(\tilde{S}, R) \trianglelefteq Q$. By Lem. A.11 we get $S_i \approx a_i.0$, and R is not both 0-closed and n-closed. If all occurrences of $z(\tilde{S}, R)$ in fact have R with a free process variable, then Q satisfies (P2) and thus is a pearl, as desired.

Otherwise, suppose Q violates (P2), i.e., it has a sub-scheme $z(\tilde{S}, R) \trianglelefteq Q$ in which R contains a shell variable and is 0-closed. Without loss of generality we may assume that $z(\tilde{S}, R)$ is an innermost such occurrence. We claim that R is a pearl. First, as a sub-scheme of Q, R must be pure, i.e., R has property (P1). Also, (P2) is easily established: since $z(\tilde{S}, R) \trianglelefteq Q$ is an *innermost* occurrence for which R is 0-closed, any $z(\tilde{S}, U) \trianglelefteq R \trianglelefteq Q$ must be such that U has a free process variable. Of course, we already know that $S_i \approx a_i.0$ in all such cases.

Finally, we claim that R satisfies (P3), i.e., every $a_i \in \mathbf{A}_n^+$ prefix in R is i-guarded; we already know that R has a shell variable. Suppose for contradiction that R has an $a_i \in \mathbf{A}_n^+$ prefix that is *not* i-guarded in R. On the other hand, $R \trianglelefteq z(\tilde{S}, R) \trianglelefteq Q$ and so, by the above, this prefix must be i-guarded in Q. This, however, implies that the sub-expression $z(\tilde{S}, R) \trianglelefteq Q$ must be i-guarded in Q. But this is impossible since any expression behind a (z, i)-action of a pearl P must be equivalent to $a_i.0$ and certainly cannot contain a variable z like $z(\tilde{S}, R)$ does. Thus, $a_i \in \mathbf{A}_n^+$ prefixes are i-guarded in R.

This proves the claim that R is a 0-closed pearl, and thus completes the last case in the proof of Lem. A.12.

∎

Although pearls do not enjoy universal invariance, as highlighted by Lem. A.12, there is an existential preservation law:

Lemma A.13 *Every 0-closed pearl P has a weak transition $P \stackrel{t}{\longrightarrow} Q$ to a 0-closed pearl Q, for some non-empty action sequence t.*

Proof: Let P be a 0-closed pearl in shell variables Z. Consider some occurrence of a sub-expression $z(\tilde{S}, U) \trianglelefteq P$ for $z \in Z$, which must exist. By Lemma A.9 and $(z, n) : z(\tilde{S}, U) \longrightarrow U$, we must have an action sequence

$$P \equiv P_0 \stackrel{\alpha_1}{\longrightarrow} P_1 \stackrel{\alpha_2}{\longrightarrow} P_2 \cdots \stackrel{\alpha_k}{\longrightarrow} P_k \equiv V\,,$$

where $k \geq 1$, $\alpha_k = (z, n)$ and V is some process substitution instance of U. Now observe that $(z, n) : P_{k-1} \longrightarrow V$ implies that $z \trianglelefteq P_{k-1}$. But then each of its predecessors P_i including P_0 must have $z \trianglelefteq P_i$, too. By Lemma A.12 applied to the first step $P \xrightarrow{\alpha_1} P_1$, we obtain that P_1 is a 0-closed pearl, or there is a sub-expression $z(\tilde{T}, Q) \trianglelefteq P_1$ in which Q is a 0-closed pearl. In the first case we are done. In the second case, we use Lemma A.9 again and get a transition sequence $s\,(z, n) : P_1 \longrightarrow W$ in which W is a (process) substitution instance of Q. However, 0-closed pearl Q has no free process variable, whence $W \equiv Q$. Overall, then, $\alpha_1 s(z, n) : P \longrightarrow Q$, as desired. ∎

Proposition A.14 *Let P be a 0-closed pearl and E be a scheme such that $P \approx E$. Then, E contains a 0-closed pearl Q.*

Proof: Let P be a 0-closed pearl in shell variables Z and $P \approx E$. First, it is easy to see that, because of this equivalence, E must be Z-pure, i.e., it is 0-closed and has at most variables $z \in Z$ and actions $a_i \in \mathbf{A}_n$. Also, E must have at least one occurrence of a shell variable from Z, and each prefix $a_i \in \mathbf{A}_n^+$ must be i-guarded. These are conditions (P1) and (P3) of pearls which are preserved under observational equivalence.

Take any occurrence $z(S_1, S_2, \ldots, S_{n-1}, R) \trianglelefteq E$ for $z \in Z$. We claim that all S_i must be process constants and weakly bisimilar to $a_i.0$. Using Lemma A.9 we get a transition sequence $t(z, i) : E \longrightarrow S_i'$ for $i = 1, \ldots, n-1$, where S_i' is some process substitution instance of S_i. This must be weakly matched by P, i.e., $t(z, i) : P \Longrightarrow P' \approx S_i'$. Because of the definition of pearls, viz. condition (P2), every successor of P after a (z, i)-action with $i < n$, such as P', is observationally equivalent to $a_i.0$, whence $P' \approx a_i.0$. Thus, $S_i' \approx a_i.0$ which implies that S_i' must be a process constant using only action a_i. Further, if S_i contains a free process variable and since S_i is recursively closed inside E (which is 0-closed), S_i is wrapped into a loop and S_i' would again contain variable z from the recursive call. Thus, $S_i \equiv S_i'$, as required.

Suppose that not every occurrence $z(\tilde{S}, R) \trianglelefteq E$ is a loop occurrence, i.e., such that R has a free process variable. Then, we find a sub-expression of E which is a 0-closed pearl as follows: take an innermost occurrence $z(\tilde{S}, R) \trianglelefteq E$ in which R is 0-closed. Then, by construction, every other $z(\tilde{V}, U) \trianglelefteq R$ inside R is such that U has a free process variable. From what was said above we know that all $V_i \approx a_i.0$ must be process constants. As a sub-scheme of E, therefore, R satisfies (P1) and (P2). We claim that R must contain a shell variable and have at most i-guarded a_i prefixes, thus satisfying (P3), which means that R is a 0-closed pearl. Guardedness is obtained in the same way as in the proof of Lem. A.12. More precisely, the fact that $z(\tilde{S}, R) \trianglelefteq E$ and all $a_i \in \mathbf{A}_n^+$ are i-guarded in E implies that they are i-guarded in R. This is simply because if the i-guard of a prefix $a_i.\, R' \trianglelefteq R \trianglelefteq E$ is in E rather than R, then all of $z(\tilde{S}, R)$ would be i-guarded in E which is impossible.

Finally, we address the question why R must have a shell variable. By contradiction, assume that the chosen innermost occurrence R is n-closed. Because of Lemma A.9, there is a transition sequence $t(z, n) : E \longrightarrow R'$, where R' is a process substitution instance of R. But R is 0-closed, and so $R' \equiv R$. The weak bisimulation $P \approx E$ then yields a sequence $t(z, n) : P \Longrightarrow P' \approx R$, i.e., the 0-closed pearl P would have a (z, n)-successor state that is n-closed. But this contradicts the structure of pearl in which each (z, n)-transition is the unfolding of a loop occurrence of z, i.e., it generates a state in which this occurrences of z are reproduced. Hence, R cannot be n-closed, i.e., R has property (P3), as desired. ∎

Proposition 4.3 [Shell Preservation]. *Let E and F be two expressions such that $E \approxeq F$. Then, E is an n-shell iff F is an n-shell.*

Proof: By symmetry, it is sufficient to establish that F is an n-shell (in shell variables Z), whenever $E \approxeq F$ and E is an n-shell (in Z). First of all, $\xi_n^Z(F) \approxeq \xi_n^Z(E) \approxeq \mathbf{A}_n$ by the definition of $\approxeq$ on schemes and since E is an n-shell. Also, E and F can have at most observable actions $a_i \in \mathbf{A}_n$ and variables from Z. In particular E and F must be 0-closed. By assumption, in addition, E contains a 0-closed pearl. In the following we show that F must contain a 0-closed pearl, too.

The 0-closed pearl of E is a Z-pure scheme $P_E \trianglelefteq E$. By Lemma A.13 we get a 0-closed pearl Q_E with $t : P_E \Longrightarrow Q_E$ for some non-empty action sequence t. Further, by Lemma A.9, there exists a transition sequence $st : E \Longrightarrow Q_E$. Note that Q_E is 0-closed and thus does not get substituted in this sequence. In other words, Q is identical to any of its process substitution instances. Since $E \approx F$, there must be a weak successor $st : F \Longrightarrow V_F$ such that $Q_E \approx V_F$. This implies that V_F contains a 0-closed pearl by Lemma A.14 and further, by iterated application of Lem. A.10 (considering that 0-closedness is preserved by transitions), we conclude that F contains a 0-closed pearl. This was to be shown.

∎

A.6 Proof of Proposition 4.4

We now proceed to prove the main factorisation result which involves various facts about instantiations and dealing with context expressions. The following decomposition and factorisation lemmas (Lems. A.16–A.19) all start from an identity of the form $\xi_n^Z(U) \equiv E$ in which an expression E is matched against a number of occurrences of instantiated shell variables as specified by U. The lemmas provide different ways of "dividing" both sides by the instantiation ξ_n^Z, essentially to express U in terms of some abstraction $U \equiv (\xi_n^Z)^{-1}(E)$ of E. This abstraction which breaks out the contexts of E_{n-1}^n from E, depends on how E is generated: by first-order substitution $E \equiv P[\tilde{F}]$ or second-order instantiation $E \equiv \theta(F)$. That identities $\xi_n^Z(U) \equiv E$ can be formally divided by ξ_n^Z, is a result of the atomicity of the contexts E_{n-1}^n.

We begin with the basic observation that a context E_{n-1}^n cannot be broken up by first-order instantiation (see also Prop. 4.1):

Lemma A.15 (Atomicity I) *Let $\tilde{E} = E_1, E_2, \ldots, E_m$ and $\tilde{U} = U_1, U_2, \ldots, U_n$ be schemes and P be a context of rank m such that $(\mu x. P)[\tilde{E}] \equiv E_{n-1}^n[\tilde{U}]$, where all E_i are free for \$i in $\mu x. P$. Then, there exist rank m contexts $\tilde{V} = V_1, V_2, \ldots, V_n$ such that $\mu x. P \equiv E_{n-1}^n[\tilde{V}]$ and $U_i \equiv V_i[\tilde{E}]$, for all $i \leq n$.*

Proof: Suppose $(\mu x. P)[\tilde{E}] \equiv E^n_{n-1}[\tilde{U}]$ as in the statement of the lemma. Let $x_1, x_2, \ldots, x_{n-1}$ be the variables bound inside the context E^n_{n-1}, in particular $x = x_1$. Since the E_i are free for $i in $\mu x. P$, the variable x_1 cannot occur free in any E_i. On the other hand, by construction, every sub-expression of $E^n_{n-1}[\tilde{U}]$ which is not a sub-expression of any U_i must contain one of $x_1, x_2, \ldots, x_{n-1}$ free. Hence, no E_i can contribute to any proper part of E^n_{n-1}. Instead, there must be rank m expressions $\tilde{V} = V_1, V_2, \ldots, V_n$ such that $U_i \equiv V_i[\tilde{E}]$ and $\mu x. P \equiv E^n_{n-1}[\tilde{V}]$. ∎

Generalising Lem. A.15 gives rise to our first decomposition lemma:

Lemma A.16 (Decomposition I) *Let P, U be expressions of rank m and 0, respectively, and U has a singleton occurrence of rank n variable z. Assume further that $P[\tilde{E}] \equiv \xi^z_n(U)$ for expressions $\tilde{E} = E_1, E_2, \ldots, E_m$, such that all E_i are free for $i in P. Then, one of the following is true:*

(i) There exists a rank m expression W with a singleton occurrence of variable z such that $\xi^z_n(W) \equiv P$ and $U \equiv W[\tilde{E}]$.

(ii) There exists an index $i = 1, \ldots, m$ and rank 1 expression V_i in which z has a singleton occurrence with $\xi^z_n(V_i) \equiv E_i$, and a rank $m+1$ expression Q such that $P \equiv Q[\$i, \$1, \$2, \ldots, \$m]$ and $U \equiv Q[V_i, \tilde{E}]$.

Proof: Note that if none of the call-back constants $\$1, \$2, \ldots, \$m$ occurs in P, then $P[\tilde{E}] \equiv P$ and case (i) holds trivially if we put $W =_{\mathrm{df}} U$. So, we assume that P actually makes a call-back. The identity $P[E] \equiv \xi^z_n(U)$ implies that z does not appear in P or any of its sub-expressions. We prove the statement of Lemma A.16 by induction on P:

- If $P \equiv \$i$, then $E_i \equiv P[\tilde{E}] \equiv \xi^z_n(U)$. If we put $V_i =_{\mathrm{df}} U$ and $Q \equiv \$1$, we can easily prove statement (ii) of the lemma.
- If $P \equiv \alpha. P'$ then the identity $\alpha. P'[\tilde{E}] \equiv P[\tilde{E}] \equiv \xi^z_n(U)$ can only hold true if $U \equiv \alpha. U'$ for some U' (in which z has a singleton occurrence) such that $P'[\tilde{E}] \equiv \xi^z_n(U')$. We now invoke the induction hypothesis with two possible outcomes:

  (i) There is a rank m expression W' with a singleton occurrence of variable z such that $\xi^z_n(W') \equiv P'$ and $U' \equiv W'[\tilde{E}]$. We define $W \equiv \alpha. W'$, for which we derive $\xi^z_n(W) \equiv \xi^z_n(\alpha. W') \equiv \alpha. \xi^z_n(W') \equiv \alpha. P' \equiv P$ and also $W[\tilde{E}] \equiv (\alpha. W')[\tilde{E}] \equiv \alpha. W'[\tilde{E}] \equiv \alpha. U' \equiv U$, which proves statement (i) of the lemma.

  (ii) There exists an index i and rank 1 expression V_i in which z has a singleton occurrence such that $\xi^z_n(V_i) \equiv E_i$ and a rank $m+1$ expression Q' such that $P' \equiv Q'[\$i, \$1, \$2, \ldots, \$m]$ and $U' \equiv Q'[V_i, \tilde{E}]$. In this case we put $Q =_{\mathrm{df}} \alpha. Q'$. With this definition it easily follows that $P \equiv Q[\$i, \$1, \$2, \ldots, \$m]$ and $U \equiv Q[V_i, \tilde{E}]$, which is statement (ii) of the lemma.

- The case that $P \equiv w(P_1, P_2, \ldots, P_m)$, where $m = \mathit{rank}(w)$, proceeds as above for prefixes.

- Suppose $P \equiv P_1 + P_2$ and $P_1[\tilde{E}] + P_2[\tilde{E}] \equiv P[\tilde{E}] \equiv \xi^z_n(U)$. This implies $U \equiv U_1 + U_2$ for some expressions U_i and $P_i[\tilde{E}] \equiv \xi^z_n(U_i)$. Since z occurs exactly once in U, only one of U_1 or U_2 can have z. Without loss of generality, assume this is U_1. Then, $P_2[\tilde{E}] \equiv \xi^z_n(U_2) \equiv U_2$. Since z is not contained in P_2, $\xi^z_n(P_2) \equiv P_2$. We apply the induction hypothesis to the equation $P_1[\tilde{E}] \equiv \xi^z_n(U_1)$ and again distinguish the two resulting cases:

  (i) There exists a rank m expression W_1 with a singleton occurrence of variable z such that $\xi^z_n(W_1) \equiv P_1$ and $U_1 \equiv W_1[\tilde{E}]$. Here, we define $W =_{\mathrm{df}} W_1 + P_2$ and calculate $\xi^z_n(W) \equiv \xi^z_n(W_1 + P_2) \equiv \xi^z_n(W_1) + P_2 \equiv P_1 + P_2 \equiv P$ and $W[\tilde{E}] \equiv (W_1 + P_2)[\tilde{E}] \equiv W_1[\tilde{E}] + P_2[\tilde{E}] \equiv U_1 + U_2 \equiv U$, which establishes condition (i) of the lemma.

  (ii) We have $\xi^z_n(V_i) \equiv E_i$ for some index i and rank 1 expression V_i in which z has a singleton occurrence, and both $P_1 \equiv Q_1[\$\mathsf{i}, \$1, \$2, \ldots, \$\mathsf{m}]$ and $U_1 \equiv Q_1[V_i, \tilde{E}]$ for some rank $m{+}1$ expression Q_1. Then, we simply define $Q =_{\mathrm{df}} Q_1 + P_2[\$2, \$3, \ldots, \$\mathsf{m}{+}1]$ and argue that

$$\begin{array}{rcl} P & \equiv & P_1 + P_2 \\ & \equiv & Q_1[\$\mathsf{i}, \$1, \$2, \ldots, \$\mathsf{m}] + P_2 \\ & \equiv & Q_1[\$\mathsf{i}, \$1, \$2, \ldots, \$\mathsf{m}] + (P_2[\$2, \$3, \ldots, \$\mathsf{m}{+}1])[\$\mathsf{i}, \$1, \$2, \ldots, \$\mathsf{m}] \\ & \equiv & (Q_1 + P_2[\$2, \$3, \ldots, \$\mathsf{m}{+}1])[\$\mathsf{i}, \$1, \$2, \ldots, \$\mathsf{m}] \\ & \equiv & Q[\$\mathsf{i}, \$1, \$2, \ldots, \$\mathsf{m}] \end{array}$$

  as well as

$$\begin{array}{rcl} Q[V_i, \tilde{E}] & \equiv & (Q_1 + P_2[\$2, \$3, \ldots, \$\mathsf{m}{+}1])[V_i, \tilde{E}] \\ & \equiv & Q_1[V_i, \tilde{E}] + (P_2[\$2, \$3, \ldots, \$\mathsf{m}{+}1])[V_i, \tilde{E}] \\ & \equiv & U_1 + P_2[\tilde{E}] \\ & \equiv & U_1 + U_2 \ \equiv \ U. \end{array}$$

  This completes the proof of statement (ii) of the lemma.

- Now let us look at the interesting case, viz. where $P \equiv \mu x.\, P'$. Here, there are two possibilities for the identity $\mu x.\, P'[\tilde{E}] \equiv P[\tilde{E}] \equiv \xi^z_n(U)$ to be true:

  Firstly, the recursion μx might have nothing to do with the instantiation ξ^z_n but is matched by U. In other words, $U \equiv \mu x.\, U'$ such that $P' \equiv \xi^z_n(U')$. In this case we proceed exactly as in the case of prefixes ($P \equiv \alpha.\, P'$) above.

  Secondly, for some expressions $\tilde{U} = U_1, U_2, \ldots, U_n$, we might have $U \equiv z(\tilde{U})$, so that the recursion $\mu x.\, P'$ stems from the occurrence of E^n_{n-1} which is instantiated for z, in particular $x = x_1$. Note that then $\xi^z_n(U) \equiv E^n_{n-1}[\tilde{U}]$ by definition of

ξ_n^z and the fact that z only occurs once in U. Specifically, $z \not\trianglelefteq U_i$ implies $\xi_n^z(\tilde{U}) \equiv \tilde{U}$. Now consider the equation

$$\mu x.\, P'[\tilde{E}] \equiv E_{n-1}^n[\tilde{U}].$$

The Atomicity Lemma A.15 gives rank m expressions $\tilde{V} = V_1, V_2, \ldots, V_n$ such that $U_i \equiv V_i[\tilde{E}]$ and $\mu x.\, P' \equiv E_{n-1}^n[\tilde{V}]$. Observe that z is not free in any V_i because it is not free in any U_i. We define $W =_{\mathrm{df}} z(\tilde{V})$ for which we easily obtain $\xi_n^z(W) \equiv \xi_n^z(z(\tilde{V})) \equiv E_{n-1}^n[\tilde{V}] \equiv \mu x.\, P' \equiv P$ and $W[\tilde{E}] \equiv z(V_1[\tilde{E}], V_2[\tilde{E}], \ldots, V_n[\tilde{E}]) \equiv U$, as required for statement (i) of the lemma.

∎

While Decomposition Lemma I (Lem. A.16) can be used to abstract out a single occurrence of a context E_{n-1}^n, there is also a decomposition lemma for multiple occurrences. For our purposes, the following special case is sufficient:

Lemma A.17 (Decomposition II) *Let P be a context of rank 1 and E, U schemes such that $\xi_n^z(U) \equiv P[E]$, where E is free for \$1 in P. Let m be the number of occurrences of \$1 in P. Then, there exists a sequence of expressions $\tilde{V} = V_1, V_2, \ldots, V_m$ and rank m expression Q such that $U \equiv Q[\tilde{V}]$, $P \equiv \xi_n^z(Q)[\$1, \ldots, \$1]$ and $E \equiv \xi_n^z(V_i)$, for all $i \leq m$.*

Proof: We prove the statement by induction on P. We will write $\$1 \trianglelefteq_m P$ to state that \$1 has exactly m individual occurrences in P, and use $\$1^m$ as an abbreviation of the sequence $\$1, \$1, \ldots, \$1$ of length m and $\$_k^m$ to denote the call-back sequence $\$k, \$(k+1), \ldots, \$m$, for $k \leq m$. Note that the degenerated case of $m = 0$ is trivially included in the statement of Lem.A.17. Therefore, we now assume that $m \geq 1$, i.e., that expression P contains \$1 and thus cannot be 0 or a variable. The remaining cases are handled as follows:

- If $P \equiv \$1$, then $m = 1$ and $E \equiv P[E] \equiv \xi_n^z(U)$. If we put $V_1 =_{\mathrm{df}} U$ and $Q \equiv \$1$, we can easily prove the statement of the Lemma.
- If $P \equiv \alpha.\, P'$, then the identity $\alpha.\, P'[E] \equiv P[E] \equiv \xi_n^z(U)$ can only hold true if $U \equiv \alpha.\, U'$ for some U' such that $P'[E] \equiv \xi_n^z(U')$. By assumption, $\$1 \trianglelefteq_m P'$. We now invoke the induction hypothesis to get m expressions $\tilde{V} = V_1, \ldots, V_m$ such that $E \equiv \xi_n^z(V_i)$, and a rank m expression Q' such that $P' \equiv \xi_n^z(Q')[\$1^m]$ and $U' \equiv Q'[\tilde{V}]$. In this case we put $Q =_{\mathrm{df}} \alpha.\, Q'$. With this definition it follows that $P \equiv \xi_n^z(Q)[\$1^m]$ and $U \equiv Q[\tilde{V}]$.
- Again, the case of $P \equiv w(P_1, P_2, \ldots, P_k)$ is treated like an action prefix.
- Suppose $P \equiv P_1 + P_2$ and $P_1[E] + P_2[E] \equiv P[E] \equiv \xi_n^z(U)$. This implies $U \equiv U_1 + U_2$ for some expressions U_i and $P_i[E] \equiv \xi_n^z(U_i)$. From $\$1 \trianglelefteq_m P$ we conclude that there are m_1, m_2 such that $m = m_1 + m_2$ and $\$1 \trianglelefteq_{m_i} P_i$. We

apply the induction hypothesis to the equations $P_i[E] \equiv \xi^z_n(U_i)$ and obtain two sequences of expressions $\tilde{V_1} = V_1, V_2, \ldots, V_{m_1}$ and $\tilde{V_2} = V_{m_1+1}, V_{m_1+2}, \ldots, V_m$ with $\xi^z_n(V_i) \equiv E$, for $i \leq m$. Further, we get $P_i \equiv \xi^z_n(Q_i)[\$1^{m_i}]$ and $U_i \equiv Q_i[\tilde{V_i}]$ for some rank m_i expression Q_i. Then, we simply define

$$Q =_{\mathrm{df}} Q_1[\$^{m_1}_1] + Q_2[\$^m_{m_1+1}]$$

and argue that

$$\begin{array}{rcl} P & \equiv & P_1 + P_2 \\ & \equiv & \xi^z_n(Q_1)[\$1^{m_1}] + \xi^z_n(Q_2)[\$1^{m_2}] \\ & \equiv & (\xi^z_n(Q_1)[\$^{m_1}_1])[\$1^m] + (\xi^z_n(Q_2)[\$^m_{m_1+1}])[\$1^m] \\ & \equiv & (\xi^z_n(Q_1)[\$^{m_1}_1] + \xi^z_n(Q_2)[\$^m_{m_1+1}])[\$1^m] \\ & \equiv & \xi^z_n(Q_1[\$^{m_1}_1] + Q_2[\$^m_{m_1+1}])[\$1^m] \\ & \equiv & \xi^z_n(Q)[\$1^m] \end{array}$$

as well as

$$\begin{array}{rcl} Q[\tilde{V}] & \equiv & (Q_1[\$^{m_1}_1] + Q_2[\$^m_{m_1+1}])[\tilde{V}] \\ & \equiv & (Q_1[\$^{m_1}_1])[\tilde{V}] + (Q_2[\$^m_{m_1+1}])[\tilde{V}] \\ & \equiv & Q_1[\tilde{V_1}] + Q_2[\tilde{V_2}] \\ & \equiv & U_1 + U_2 \ \equiv \ U. \end{array}$$

- Now take the recursion case where $P \equiv \mu x.\, P'$. Here, there are two possibilities for the identity $\mu x.\, P'[E] \equiv P[E] \equiv \xi^z_n(U)$ to be true:

  Firstly, the recursion μx is not matched by the instantiation ξ^z_n but by U. In other words, $U \equiv \mu x.\, U'$ such that $P' \equiv \xi^z_n(U')$. In this case we proceed exactly as in the case of prefixes ($P \equiv \alpha.\, P'$) above.

  Secondly, for some schemes $\tilde{U} = U_1, U_2, \ldots, U_n$ we might have $U \equiv z(\tilde{U})$, so that the recursion $\mu x.\, P'$ stems from the occurrence of E^n_{n-1} which is instantiated for z. Note that then $\xi^z_n(U) \equiv E^n_{n-1}[\xi^z_n(\tilde{U})]$. Now consider the equation

$$\mu x.\, P'[E] \equiv E^n_{n-1}[\xi^z_n(\tilde{U})].$$

  Because of the Atomicity Lemma (Lem. A.15), there must be rank 1 expressions $\tilde{P} = P_1, P_2, \ldots, P_n$ such that $\xi^z_n(U_i) \equiv P_i[E]$ and $\mu x.\, P' \equiv E^n_{n-1}[\tilde{P}]$. Let m_i be the number of occurrences of \$1 in P_i, i.e., $\$1 \trianglelefteq_{m_i} P_i$. We must have $\sum_{i=1}^n m_i = m$. Now we apply the induction hypothesis to each identity $\xi^z_n(U_i) \equiv P_i[E]$ and obtain, for each $i \leq n$, a rank m_i expression Q_i and m_i expressions $\tilde{V_i} = V_{k_i+1}, V_{k_i+2}, \ldots, V_{k_i+m_i}$ with the offset $k_i =_{\mathrm{df}} \sum_{j=1}^{i-1} m_j$ such that $U_i \equiv Q_i[\tilde{V_i}]$, $P_i \equiv \xi^z_n(Q_i)[\$1^{m_i}]$ and $E \equiv \xi^z_n(V_i)$, for all $i \leq m$. Note that $k_1 = 0$, and for any expression T of rank m_i we have $T[\tilde{V_i}] \equiv T[\$^{k_i+m_i}_{k_i+1}][\tilde{V}]$. We define

$$Q =_{\mathrm{df}} z(Q_1[\$^{k_1+m_1}_{k_1+1}], Q_2[\$^{k_2+m_2}_{k_2+1}], \ldots, Q_n[\$^{k_n+m_n}_{k_n+1}])$$

for which one computes

$$
\begin{aligned}
Q[\tilde{V}] &\equiv (z(Q_1[\$_{k_1+1}^{k_1+m_1}], Q_2[\$_{k_2+1}^{k_2+m_2}], \ldots, Q_n[\$_{k_n+1}^{k_n+m_n}]))[\tilde{V}] \\
&\equiv z(Q_1[\$_{k_1+1}^{k_1+m_1}][\tilde{V}], Q_2[\$_{k_2+1}^{k_2+m_2}][\tilde{V}], \ldots, Q_n[\$_{k_n+1}^{k_n+m_n}][\tilde{V}]) \\
&\equiv z(Q_1[\tilde{V}_1], Q_2[\tilde{V}_2], \ldots, Q_n[\tilde{V}_n]) \\
&\equiv z(U_1, U_2, \ldots U_n) \equiv U
\end{aligned}
$$

and also

$$
\begin{aligned}
&\xi_n^z(Q)[\$1^m] \\
&\equiv \xi_n^z(z(Q_1[\$_{k_1+1}^{m_1}], Q_2[\$_{k_2+1}^{k_2+m_2}], \ldots, Q_n[\$_{k_n+1}^{k_n+m_n}]))[\$1^m] \\
&\equiv E_{n-1}^n[\xi_n^z(Q_1)[\$_{k_1+1}^{k_1+m_1}], \xi_n^z(Q_2)[\$_{k_2+1}^{k_2+m_2}], \ldots, \xi_n^z(Q_n)[\$_{k_n+1}^{k_n+m_n}]][\$1^m] \\
&\equiv E_{n-1}^n[\xi_n^z(Q_1)[\$_{k_1+1}^{k_1+m_1}][\$1^m], \xi_n^z(Q_2)[\$_{k_2+1}^{k_2+m_2}][\$1^m], \xi_n^z(Q_n)[\$_{k_n+1}^{k_n+m_n}][\$1^m]] \\
&\equiv E_{n-1}^n[\xi_n^z(Q_1)[\$1^{m_1}], \xi_n^z(Q_2)[\$1^{m_2}], \ldots, \xi_n^z(Q_n)[\$1^{m_n}]] \\
&\equiv E_{n-1}^n[P_1, P_2, \ldots, P_n] \equiv \mu x.\, P' \equiv P,
\end{aligned}
$$

which was to be shown.

∎

Let $\tilde{S} = S_1, S_2, \ldots, S_n$ and $\tilde{T} = T_1, T_2, \ldots, T_n$ be two sequences of process schemes. We say $\tilde{S}$ and $\tilde{T}$ are k-*compatible* if there exists a sequence $\tilde{C} = C_1, C_2, \ldots, C_n$ of rank k expressions and schemes $\tilde{R}_j = R_{j1}, R_{j2}, \ldots, R_{jk}$ (where $j = 1, 2$) such that $S_i \equiv C_i[\tilde{R}_1]$ and $T_i \equiv C_i[\tilde{R}_2]$. Observe that two sequences of length n are k-compatible whenever $k \geq n$. Then, the statement is trivial since we can put $C_i =_{\mathrm{df}}$ \$i, $R_{1i} =_{\mathrm{df}} S_i$ and $R_{2i} =_{\mathrm{df}} T_i$, for $i = 1, \ldots, n$. Compatibility is stronger the smaller we choose k to be. For $k = 0$ we get identity: $\tilde{S}$ and $\tilde{T}$ are 0-compatible iff $S_i \equiv T_i$ for all $i = 1, \ldots, n$.

We say that (occurrences of) rank n variables z_1 and z_2 are k-*compatible in* expression E if, for any two occurrences $z_1(\tilde{S}) \trianglelefteq E$ and $z_2(\tilde{T}) \trianglelefteq E$, the sequences $\tilde{S}$ and $\tilde{T}$ are k-compatible. This happens, for instance, if both occurrences arise from the instantiation of the same rank k variable w in E, e.g., $z_1(\tilde{S}) \trianglelefteq \theta(w)[\tilde{F}] \trianglelefteq E$ and $z_2(\tilde{T}) \trianglelefteq \theta(w)[\tilde{E}] \trianglelefteq E$. This is a side result of the following factorisation lemma which will become important later.

Lemma A.18 (Factorisation) *Let E be a scheme of maximal recursion depth rd and in which all free variables have maximal rank rk. Suppose rd < 2 and rk $< n-2$, or rd $< n-1$ and rk < 2. Suppose $\theta(E) \equiv \xi_n^z(U)$ for some instantiation θ and scheme U in which rank n variable z has a singleton loop occurrence. Then, there exists a scheme U' together with an instantiation θ' such that*

(i) $\theta = \xi_n^z \circ \theta'$

(ii) $\theta'(E) \equiv U'\{z/z'\}$

(iii) $U \equiv \xi_n^{z'}(U')$,

where z' is a fresh rank n variable not appearing in E or U. Moreover, all occurrences of z in $\theta'(E)$ are rk-compatible.

Proof: Let E, θ, U, z be given as stated and Y be the set of free variables occurring in E. It will be convenient to abuse notation and write the instantiation $\xi_n^{z'}(U')$ more explicitly as $U'\{E_{n-1}^n/z'\}$ in the following. The literal application of this substitution would turn $z'(\tilde{F})$ into $E_{n-1}^n(\tilde{F})$ which is not strictly a well-formed expression. Of course, we mean the proper (meta-level) instantiation $E_{n-1}^n[\tilde{F}]$.

Since z occurs only once in U we must have

$$\xi_n^z(U) \equiv U\{E_{n-1}^n/z\} \equiv u\{E_{n-1}^n[U_1, \ldots, U_{n-1}, U_n]/y\}$$

for some schemes U_i and u such that $U \equiv u\{E_{n-1}^n(U_1, \ldots, U_{n-1}, U_n)/y\}$. Moreover, since the occurrence of z is a loop occurrence, U_n has a free process variable, say y^*, and this occurrence of variable y^* is bound inside a recursion μy^* in $\xi_n^z(U)$ and thus also in $\theta(E)$. There are two ways in which y^* and its recursion μy^* may appear in $\theta(E)$. First, both may be introduced through the instantiation θ or they may be part of expression E. It is not possible that the occurrence of y^* is in E and its associated recursion μy^* part of θ, or vice versa. The reason is simply that θ is free, i.e., variables cannot be captured by instantiation. This means that, if expression E has recursion depth < 2, then every sub-expression of $E' \trianglelefteq E$ in which y^* is free must be recursion-free. Otherwise, E would contain one recursion nested inside another, i.e., have recursion depth ≥ 2. We will use this fact in the following proof. Also, freeness of θ means that y^* is not in the range of θ which amounts to the statement that, if $y^* \trianglelefteq \theta(E')$ for any sub-expression $E' \trianglelefteq E$, then $y^* \trianglelefteq E'$.

Let θ be an instantiation and w be a variable. An instantiation θ_w is a w-*variant* of θ if it has the same domain as θ and coincides with θ on all variables save possibly w, i.e., $\theta(z) \equiv \theta_w(z)$ for all $z \neq w$. We will obtain *rk*-compatibility of all z in $\theta'(E)$ directly from the special construction of θ'. More precisely, the factorising instantiation θ' will be a variant of θ which is exactly like θ, except for one single variable $w^* \in Y$ (of some rank m) where $\theta'(w^*)$ introduces a singleton occurrence of z. For all other $w \in Y$ with $w \neq w^*$, $\theta'(w) \equiv \theta(w)$. The singleton occurrence of z in $\theta'(w^*)$ corresponds to a unique sub-expression $z(\tilde{C}) \trianglelefteq \theta'(w^*)$ of rank (at most) *rk* for some $\tilde{C} = C_1, C_2, \ldots, C_n$. Obviously, every occurrence of $z(\tilde{S}) \trianglelefteq \theta'(w^*)[\tilde{D}]$ (there is exactly one!) is of the form $\tilde{S} \equiv \tilde{C}[\tilde{D}]$. This lifts to arbitrary expressions E in which z is not contained and to instantiations θ' in which a singleton z is introduced by θ' for a unique variable w^*, i.e., $z \trianglelefteq \theta'(y)$ implies $y = w^*$ and $z \trianglelefteq_1 \theta(w^*)$. In this situation one proves, by induction on E, that each occurrence $z(\tilde{S}) \trianglelefteq \theta'(E)$ is obtained from the same $\tilde{C}$ by instantiation, i.e., there are $\tilde{R} = R_1, R_2, \ldots, R_k$ such that $S_i \equiv C_i[\tilde{R}]$. The number k is bounded by the maximal rank *rk* of any variable in E. This is a stronger form of "global" compatibility from which the simple compatibility of z in $\theta'(E)$ which is stated in Lemma A.18 follows. As an aside, the notion "global" is justified since a single context family $\tilde{E}$ can be found which is good for all occurrences $z(\tilde{S})$, as opposed to the "local" notion in which each pair $z(\tilde{S})$ and $z(\tilde{T})$ possesses a possibly different compatibility context.

Note that, although $z(\tilde{C})$ appears exactly once in $\theta'(w^*)$ there may be many occurrences $z(\tilde{S}) \trianglelefteq \theta'(E)$, depending on how many times the variable w^* appears in E. Note further that $\theta(E) \equiv \xi^z_n(U)$ implies that variable z cannot be contained in $\theta(E)$, and thus neither in any $\theta(y)$ such that $y \in Y$. This means, in particular, that $\xi^z_n(\theta(y)) \equiv \theta(y)$ for all $y \in Y$. Then, if θ' is a w-variant of θ, we have the desired equation $\theta = \xi^z_n \circ \theta'$ as soon as $\xi^z_n(\theta'(w)) \equiv \theta(w)$. We will also construct θ' so that z' does not appear in its image, i.e., $z' \not\trianglelefteq \theta'(y)$ for $y \in Y$.

The following proof of Lem. A.18 is by induction on E. Observe that the cases $E \equiv$ \$k are impossible since $E^n_{n-1}[U_1, U_2, \ldots, U_{n-1}, U_n]$ is a sub-term of $\theta(E)$:

- Suppose $E \equiv x$ and $\theta(x) \equiv \theta(E) \equiv \xi^z_n(U)$. Define θ' as an x-variant of θ such that $\theta'(x) =_{\mathrm{df}} U$ and $\theta'(y) = \theta(y)$ for all other variables $y \in Y \setminus \{x\}$. Then, trivially, $(\xi^z_n \circ \theta')(x) = \xi^z_n(U) = \theta(x)$ by definition, and $(\xi^z_n \circ \theta')(y) = \xi^z_n(\theta(y)) = \theta(y)$ for $y \in Y \setminus \{x\}$ since $z \not\trianglelefteq \theta(y)$. This proves statement (i) of the lemma.

  Next, let $U' =_{\mathrm{df}} U$. By assumption $z \trianglelefteq U'$. On the other hand, $z' \not\trianglelefteq U$, so we get $U'\{z/z'\} \equiv U\{z/z'\} \equiv U \equiv \theta'(x) \equiv \theta'(E)$, which is statement (ii).

Finally, (iii) holds because $U \equiv U\{E^n_{n-1}/z'\} \equiv U'\{E^n_{n-1}/z'\}$.

- If $E \equiv \alpha.F$, then $\alpha.\theta(F) \equiv \theta(E) \equiv \xi^z_n(U)$. Since E^n_{n-1} always starts with a recursion, U cannot be of the form $z(U')$. Instead, there must exist V with $U \equiv \alpha.V$ such that z has a singleton loop occurrence in V and $\theta(F) \equiv \xi^z_n(V)$. We can now apply the induction hypothesis which gives an expression V' in which z occurs and $V \equiv V'\{E^n_{n-1}/z'\}$, together with an instantiation θ' such that $V'\{z/z'\} \equiv \theta'(F)$ and $\theta = \xi^z_n \circ \theta'$. This is already statement (i) of the lemma in this case.

  Now take $U' =_{\mathrm{df}} \alpha.V'$ for which

  $$U'\{E^n_{n-1}/z'\} \equiv (\alpha.V')\{E^n_{n-1}/z'\} \equiv \alpha.V'\{E^n_{n-1}/z'\} \equiv \alpha.V \equiv U$$

  as required by statement (ii). Note that $z \trianglelefteq U'$ since it features in V'.

  Furthermore,

  $$U'\{z/z'\} \equiv (\alpha.V')\{z/z'\} \equiv \alpha.V'\{z/z'\} \equiv \alpha.\theta'(F) \equiv \theta'(\alpha.F) \equiv \theta'(E).$$

  This proves (iii) and completes the case.

- If $E \equiv E_1 + E_2$ then $\theta(E_1) + \theta(E_2) \equiv \theta(E) \equiv \xi^z_n(U)$. Again, recall that any expression E^n_{n-1} starts with a recursion, whence U cannot be of the form $z(T)$ but must be $U \equiv U_1 + U_2$. Moreover, $\theta(E_i) \equiv \xi^z_n(U_i)$. By assumption, z appears exactly once in $U_1 + U_2$, which is a loop occurrence. Without loss of generality, let z occur in U_1 but not in U_2, i.e., $\xi^z_n(U_2) \equiv U_2$. The induction hypothesis yields an expression U_1' such that $U_1 \equiv U_1'\{E^n_{n-1}/z'\}$ and an instantiation θ' with $\theta = \xi^z_n \circ \theta'$ and $U_1'\{z/z'\} \equiv \theta'(E_1)$.

  We define $U' =_{\mathrm{df}} U_1' + \theta'(E_2)\{z'/z\}$ for which we find that

  $$\begin{array}{rcl} U'\{E^n_{n-1}/z'\} & \equiv & U_1'\{E^n_{n-1}/z'\} + \theta'(E_2)\{z'/z\}\{E^n_{n-1}/z'\} \\ & \equiv & U_1 + \theta'(E_2)\{E^n_{n-1}/z\} \\ & \equiv & U_1 + (\xi^z_n \circ \theta')(E_2) \\ & \equiv & U_1 + \theta(E_2) \\ & \equiv & U_1 + U_2 \;\equiv\; U \end{array}$$

  and similarly

  $$U'\{z/z'\} \equiv U_1'\{z/z'\} + \theta'(E_2)\{z'/z\}\{z/z'\} \equiv \theta'(E_1) + \theta'(E_2) \equiv \theta'(E),$$

  given that $z' \not\trianglelefteq \theta'(E)$. Thus we have proved statements (i)–(iii) of the lemma.

- If $E \equiv w(\tilde{D})$ for a $\tilde{D} = D_1, D_2, \ldots, D_m$, then $\xi^z_n(U) \equiv \theta(E) \equiv \theta(w)[\theta(\tilde{D})]$, where $\theta(\tilde{D})$ abbreviates the sequence $\theta(D_1), \theta(D_2), \ldots, \theta(D_m)$. By the Decomposition Lemma (Lem. A.16), two cases can arise which are handled separately in the following:

(i) The first case to consider is where we have a rank m expression W in which z has a singleton loop occurrence and $\theta(w) \equiv \xi^z_n(W)$ and $U \equiv W[\theta(\tilde{D})]$. Define θ' as the w-variant of θ such that $\theta'(w) =_{\mathrm{df}} W$ and let $U' =_{\mathrm{df}} W[\theta'(\tilde{D})\{z'/z\}]$, where the sequence $\theta'(D_1)\{z'/z\}, \theta'(D_2)\{z'/z\}, \ldots, \theta'(D_m)\{z'/z\}$ is abbreviated as $\theta'(\tilde{D})\{z'/z\}$. Then, it is easy to see that $\xi^z_n \circ \theta' = \theta$ and

$$\begin{aligned} U'\{z/z'\} &\equiv (W[\theta'(\tilde{D})\{z'/z\}])\{z/z'\} \\ &\equiv (W\{z/z'\})[\theta'(\tilde{D})\{z'/z\}\{z/z'\}] \\ &\equiv W[\theta'(\tilde{D})] \\ &\equiv \theta'(w)[\theta'(\tilde{D})] \;\equiv\; \theta'(E)\,, \end{aligned}$$

observing that $z' \not\trianglelefteq \theta'(\tilde{D})$, and quite analogously

$$\begin{aligned} U'\{E^n_{n-1}/z'\} &\equiv (W[\theta'(\tilde{D})\{z'/z\}])\{E^n_{n-1}/z'\} \\ &\equiv W[\theta'(\tilde{D})\{z'/z\}\{E^n_{n-1}/z'\}] \\ &\equiv W[\theta'(\tilde{D})\{E^n_{n-1}/z\}] \\ &\equiv W[(\xi^z_n \circ \theta')(\tilde{D})] \\ &\equiv W[\theta(\tilde{D})] \;\equiv\; U. \end{aligned}$$

(ii) In the second case, there is an index i and expression V_i with $\theta(D_i) \equiv \xi^z_n(V_i)$, where z has a singleton loop occurrence in V_i. Moreover, there must exist a rank $m+1$ expression Q such that $\theta(w) \equiv Q[\$\mathsf{i}, \$1, \$2, \ldots, \$\mathsf{m}]$ and $U \equiv Q[V_i, \theta(\tilde{D})]$.

Starting from $\theta(D_i) \equiv \xi^z_n(V_i)$ we may invoke the induction hypothesis for D_i. This gives an expression V'_i and instantiation θ' such that $V_i \equiv V'_i\{E^n_{n-1}/z'\}$, $\theta'(D_i) \equiv V'_i\{z/z'\}$ and $\theta = \xi^z_n \circ \theta'$.

On the one hand, we have $\theta(w) \equiv \xi^z_n(\theta'(w)) \equiv \theta'(w)\{E^n_{n-1}/z\}$. On the other hand, we have $\theta(w) \equiv Q[\$\mathsf{i}, \$1, \$2, \ldots, \$\mathsf{m}]$ such that $U \equiv Q[V_i, \theta(\tilde{D})]$. To the equivalence

$$\xi^z_n(\theta'(w)) \;\equiv\; \theta'(w)\{E^n_{n-1}/z\} \;\equiv\; Q[\$\mathsf{i}, \$1, \$2, \ldots, \$\mathsf{m}]$$

we can apply the Decomposition Lemma (Lem. A.16). Since for none of the callback expressions $E_i \in \{\$1, \$2, \ldots, \$\mathsf{m}\}$ there can be a V_i in which z appears and with $\xi^z_n(V_i) \equiv E_i$, only the first option (i) of the statement of the Decomposition Lemma (Lem. A.16) applies. This obtains a rank $m+1$ expression Q' (with singleton z) such that $Q'[\$\mathsf{i}, \$1, \$2, \ldots, \$\mathsf{m}] \equiv Q$ and $Q'\{E^n_{n-1}/z\} \equiv \xi^z_n(Q)$. Note that $Q'\{z/z'\} \equiv Q'$ as z' is fresh, and thus $z' \not\trianglelefteq Q'$ by construction. The same holds for $\theta'(\tilde{D})$, i.e., none of $\theta'(\tilde{D}_i)$ has z' free. Then, let

$U' =_{\rm df} Q'\{z'/z\}[V_i', \theta'(\tilde{D})\{z'/z\}]$. We compute

$$
\begin{array}{rcl}
U'\{z/z'\} & \equiv & (Q'\{z'/z\}[V_i', \theta'(\tilde{D})\{z'/z\}])\{z/z'\} \\
& \equiv & Q'\{z'/z\}\{z/z'\}[V_i'\{z/z'\}, \theta'(\tilde{D})\{z'/z\}\{z/z'\}] \\
& \equiv & Q'\{z'\{z/z'\}/z\}\{z/z'\}[V_i'\{z/z'\}, \theta'(\tilde{D})\{z'/z\}\{z/z'\}] \\
& \equiv & Q'\{z/z\}\{z/z'\}[V_i'\{z/z'\}, \theta'(\tilde{D})\{z'/z\}\{z/z'\}] \\
& \equiv & Q'\{z/z'\}[V_i'\{z/z'\}, \theta'(\tilde{D})\{z/z'\}] \\
& \equiv & Q'[\theta'(D_i), \theta'(\tilde{D})] \\
& \equiv & (Q'[\$\mathsf{i}, \$1, \$2, \ldots, \$\mathsf{m}])[\theta'(\tilde{D})] \\
& \equiv & \theta'(w)[\theta'(\tilde{D})] \\
& \equiv & \theta'(w(\tilde{D})) \;\equiv\; \theta'(E).
\end{array}
$$

Further,

$$
\begin{array}{rl}
U'\{E^n_{n-1}/z\} & \\
\equiv & (Q'\{z'/z\}[V_i', \theta'(\tilde{D})\{z'/z\}])\{E^n_{n-1}/z'\} \\
\equiv & Q'\{z'/z\}\{E^n_{n-1}/z'\}[V_i'\{E^n_{n-1}/z'\}, \theta'(\tilde{D})\{z'/z\}\{E^n_{n-1}/z'\}] \\
\equiv & Q'\{E^n_{n-1}/z'\}\{E^n_{n-1}/z\}[V_i'\{E^n_{n-1}/z'\}, \theta'(\tilde{D})\{E^n_{n-1}/z'\}\{E^n_{n-1}/z\}] \\
\equiv & Q'\{E^n_{n-1}/z\}[V_i, \theta'(\tilde{D})\{E^n_{n-1}/z\}] \\
\equiv & Q'\{E^n_{n-1}/z\}[V_i, (\xi^z_n \circ \theta')(\tilde{D})] \\
\equiv & Q[V_i, \theta(\tilde{D})] \;\equiv\; U\,,
\end{array}
$$

as desired.

- If $E \equiv \mu x.\, F$, then $\mu x.\, \theta(F) \equiv \theta(E) \equiv \xi^z_n(U)$. We distinguish two cases:

  The simplest scenario for this is where $U \equiv \mu x.\, V$ and $\theta(F) \equiv \xi^z_n(V)$. Here, we proceed by induction hypothesis which yields a V' (with singleton occurrence z) such that $V = V'\{E^n_{n-1}/z'\}$, together with a θ' so that $\theta = \xi^z_n \circ \theta'$ and $\theta'(F) \equiv V'\{z/z'\}$. If we define $U' =_{\rm df} \mu x.\, V'$, we can easily verify that $U'\{z/z'\} \equiv (\mu x.\, V')\{z/z'\} \equiv \mu x.\, V'\{z/z'\} \equiv \mu x.\, \theta'(F) \equiv \theta'(E)$ and $U'\{E^n_{n-1}/z'\} \equiv \mu x.\, V'\{E^n_{n-1}/z'\} \equiv \mu x.\, V \equiv U$.

  More tricky is the scenario in which the recursion $\mu x.\, F$ matches the top-level recursion μx_1 of the context expression E^n_{n-1}. In this case, $U \equiv z(\tilde{U})$ and $\mu x.\, \theta(F) \equiv E^n_{n-1}[\tilde{U}]$, for some schemes $\tilde{U} = U_1, U_2, \ldots, U_n$. In other words,

  $$U_1 + \tau.\, E^n_{n-2}[U_2, \ldots, U_{n-1}, U_n, x_1] \;\equiv\; \theta(F). \tag{7}$$

  This is the situation where scheme E peels off the top-level recursion of E^n_{n-1} associated with the occurrence of z that we are interested in. We claim that this is not possible under our syntactic restrictions on expression E.

Let us consider what equation (7) implies for F. First, observe that due to (7), the expression $\theta(F)$ must have at least two free process variables, viz. the variable x_1 and also y^* which is free in t by assumption (and distinct from x_1). On the other hand, θ is free, so x_1, y^* must be contained in F already and not in the range of θ. But if y^* is free in F, then it is also free in E. From what has been said above we conclude that E must have recursion depth *rd* ≥ 2. Otherwise, E in which y^* is free could not contain another recursion μx_1. By assumption, then, E has recursion depth *rd* $< n-1$ and all free variables of E (and thus of F) have rank *rk* < 2. We now claim that F must contain all the recursions $\mu x_2, \ldots, \mu x_{n-1}$ inside E^n_{n-2}.

Note that F cannot itself be a process variable since it must have *both* x_1 and y^* free. In fact, F can only be the application of a rank 1 variable or a summation. It turns out, as we are now going to show, that F must be a summation, i.e., $F \equiv T_1 + F_2$, since the term $E^n_{n-2}[U_2, \ldots, U_{n-1}, U_n, x_1]$ to be matched inside $\theta(F)$ has two free variables, y^* and x_1. For contradiction, suppose otherwise, i.e., that the sum arises from the substitution of a rank 1 variable $F \equiv w(G')$ so that $U_1 + E^n_{n-2}[U_2, \ldots, U_{n-1}, U_n, x_1] \equiv \theta(w)[\theta(G')]$. The free variables x_1, y^* of $\theta(w)[\theta(G')]$ are the free variables of the call-backs $\theta(G')$ as θ is free. This implies that all occurrences of y^* and x_1 inside $E^n_{n-2}[U_2, \ldots, U_{n-1}, U_n, x_1]$ must be simultaneously covered by (i.e., inside a sub-term of) possibly multiple occurrences of $\theta(G')$. Because of the way E^n_{n-2} is constructed, this cannot hold true unless the variable x_{n-1} bound innermost in E^n_{n-2}, too, is contained in $\theta(G')$. However, this is impossible since the call-back $\theta(G')$ of $\theta(w)[\theta(G')]$ cannot have any variable free which is bound inside $\theta(w)$.

This proves that $F \equiv T_1 + F_2$ with $\theta(T_1) \equiv U_1$ and

$$\begin{aligned}\theta(F_2) &\equiv E^n_{n-2}[U_2, \ldots, U_{n-1}, U_n, x_1] \\ &\equiv \mu x_2.\, (U_2 + \tau.\, E^n_{n-3}[U_3, \ldots, U_{n-1}, U_n, x_1, x_2]).\end{aligned}$$

We are now in a similar situation as above: we must consider the ways in which the recursion μx_2 is generated by $\theta(F_2)$. Again, because of the two free variables y^*, x_1 in $E^n_{n-2}[U_2, \ldots, U_{n-1}, U_n, x_1]$, the cases $F_2 \equiv x$ and $F_2 \equiv w(G')$ are excluded. Hence, $F_2 \equiv \mu x_2.\, F'$. We may now continue in this way, inductively, to show that all the recursions $\mu x_1, \mu x_2, \ldots, \mu x_{n-1}$ and associated summations $U_i + \tau.\, \mu x_{i+1}. \cdots$ of E^n_{n-1} must be part of E. More precisely, there must be expressions T_1, T_2, ..., T_n such that $\theta(T_i) \equiv U_i$ and $E \equiv \mu x_1.\, F \equiv E^n_{n-1}[T_1, T_2, \ldots, T_n]$. This, however, is impossible because of the assumption that E has recursion depth *rd* smaller than $n-1$ (which is the number of recursions inside E^n_{n-1}).

∎

Lemma A.19 (Pearl Abstraction) *Let P be a pearl with shell variables Z, each of which has a singleton occurrence in P. Suppose, $\xi_n^Y(P) \equiv \xi_n^{z'}(U)$ for some proper subset $Y \subset Z$, scheme U and fresh rank n variable $z' \notin Z$, such that all occurrences of z' in U are rk-compatible in U (for rk $< n-2$) with some shell variable $z \in Z \setminus Y$ also appearing in U. Then, there exists a partitioning $Y = X \uplus Z'$ such that $U \equiv \xi_n^X(P\{z'/Z'\})$ where $P\{z'/Z'\}$ is pearl P with all variables Z' renamed to z'.*

Proof: The proof is by induction on the structure of P and heavily relies on the observations made in Lem. A.4. Notice that the identity $\xi_n^Y(P) \equiv \xi_n^{z'}(U)$ implies that $z' \not\trianglelefteq P$ and $\forall w \in Y.\, w \not\trianglelefteq U$. In addition, the shell variables $z \in Z \setminus Y$ must all occur in U. We shall say that z' has an *essential* occurrence in U to express that z' is *rk*-compatible to some such shell variable $z \in Z \setminus Y$. Further, we observe that, if U does not contain the variables z', then putting $X =_{\mathrm{df}} Y$ and $Z' =_{\mathrm{df}} \emptyset$ solves the problem trivially. Hence, we may assume $z' \trianglelefteq U$ in the following. Also, since P is a pearl, it must contain at least one occurrence of a shell variable from $Z \neq \emptyset$, and thus cannot be a process variable or a call-back constant.

- If $P \equiv \alpha.\, P'$, then the identity $\xi_n^Y(P) \equiv \xi_n^{z'}(U)$ forces U to be of the form $U \equiv \alpha.\, U'$ and $\xi_n^Y(P') \equiv \xi_n^{z'}(U')$. It is clear that P' is again a pearl in singleton shell variables Z. By induction hypothesis, there is a partitioning $Y = X \uplus Z'$ such that $U' \equiv \xi_n^X(P'\{z'/Z'\})$. From this, $U \equiv \alpha.\, U' \equiv \alpha.\, \xi_n^X(P'\{z'/Z'\}) \equiv \xi_n^X(\alpha.\, P'\{z'/Z'\}) \equiv \xi_n^X(P\{z'/Z'\})$ follows trivially.

- If $P \equiv P_1 + P_2$, then $\xi_n^Y(P) \equiv \xi_n^{z'}(U)$ means that U is of the form $U \equiv U_1 + U_2$ and $\xi_n^Y(P_i) \equiv \xi_n^{z'}(U_i)$. Since all Y are singletons in P, there is a split $Y = Y_1 \uplus Y_2$ where Y_1 and Y_2 are shell variables from Y occurring in P_1 and P_2, respectively, so that $\xi_n^Y(P_i) \equiv \xi_n^{Y_i}(P_i)$, for $i = 1, 2$.

  Suppose P_i contains some shell variables from Z. Then, P_i must be a pearl with singleton shell variables Z. We can apply the induction hypothesis to the identity $\xi_n^{Y_i}(P_i) \equiv \xi_n^{z'}(U_i)$, noting that $z' \notin Y_i$. This yields a partitioning $Y_i = X_i \uplus Z'_i$ and $U_i \equiv \xi_n^{X_i}(P_i\{z'/Z'_i\})$.

  If P_i has no Z, i.e., $P_i \equiv \xi_n^{Y_i}(P_i) \equiv \xi_n^{z'}(U_i)$, we get the same result albeit with a different argument, observing that then variable z' cannot occur in U_i. By way of contradiction assume otherwise, i.e., $z' \trianglelefteq U_i$. Then, the expression $\xi_n^{z'}(U_i)$, in which z' is by assumption an essential occurrence, must possess — together with z' — a prefix for at least one shell action a_j, where $j \geq 1$. This prefix then occurs in $P_i \trianglelefteq P$ and, since P is a pearl, must be j-guarded by an occurrence of some shell variable $w \in Z$. Since $P \equiv P_1 + P_2$, the associated guard occurrence of w would have to occur in P_i already. However, by assumption, P_i does not have any Z free, which is a contradiction. Thus, we find that whenever P_i does not contain shell variables from Z, then $z' \not\trianglelefteq U_i$ and, therefore, $P_i \equiv \xi_n^{Y_i}(P_i) \equiv \xi_n^{z'}(U_i) \equiv U_i$. We can then define $X_i =_{\mathrm{df}} \emptyset$ and $Z'_i =_{\mathrm{df}} \emptyset$, and trivially obtain $U_i \equiv \xi_n^{X_i}(P_i\{z'/Z'_i\})$ as in the case where P_i is a pearl.

We now conclude the proof in this step of the induction as follows: putting $X =_{\rm df} X_1 \cup X_2$ and $Z' =_{\rm df} Z'_1 \cup Z'_2$ we get $Y = Y_1 \uplus Y_2 = (X_1 \uplus Z'_1) \uplus (X_2 \uplus Z'_2) = (X_1 \uplus X_2) \uplus (Z'_1 \uplus Z'_2) = (X_1 \cup X_2) \uplus (Z'_1 \cup Z'_2) = X \uplus Z'$ and, further, $U \equiv U_1 + U_2 \equiv \xi_n^{X_1}(P_1\{z'/Z'_1\}) + \xi_n^{X_2}(P_2\{z'/Z'_2\}) \equiv \xi_n^{X}(P_1\{z'/Z'_1\}) + \xi_n^{X}(P_2\{z'/Z'_2\}) \equiv \xi_n^{X}(P_1\{z'/Z'_1\} + P_2\{z'/Z'_2\}) \equiv \xi_n^{X}(P_1\{z'/Z'\} + P_2\{z'/Z'\}) \equiv \xi_n^{X}((P_1 + P_2)\{z'/Z'\}) \equiv \xi_n^{X}(P\{z'/Z'\})$. These calculations exploit that none of the variables in $Y_i = X_i \uplus Z'_i$ is contained in P_j for $i \neq j$.

- Let $P \equiv w(\tilde{S})$, for $\tilde{S} = S_1, S_2, \ldots, S_m$. Since P is a pearl and thus pure in Z we must have $w \in Z$ and $m = n$. Because each Z has at most a singleton occurrence in P and w can no longer be free in any S_i, there must be a partition $Y = Y_1 \uplus Y_2 \uplus \cdots \uplus Y_n \uplus \{w\}$ with $\xi_n^{Y}(\tilde{S}_i) \equiv \xi_n^{Y_i}(\tilde{S}_i)$. In particular, note that $\forall i \leq n.\, w \notin Y_i$.

  To begin with, we show that whenever $\xi_n^{z'}(V_i) \equiv \xi_n^{Y_i}(S_i)$ and z' has an essential occurrence in V_i, then $i = n$ and S_i is again a pearl in shell variables Z_i. If $z' \trianglelefteq V_i$ we know, because of the essential nature of the occurrence z', that $\xi_n^{z'}(V_i)$ must possess prefixes for at least two distinct actions, say a_{j_1} and a_{j_2}, where $j_1, j_2 \in \{1, 2, \ldots, n-1\}$ and $j_1 \neq j_2$. Due to the identity $\xi_n^{Y_i}(S_i) \equiv \xi_n^{z'}(V_i)$, these prefixes must appear in $S_i \trianglelefteq w(\tilde{S}) \equiv P$. By way of contradiction assume that $i \leq n-1$. Recall that P is a pearl and $w \in Y$ a shell variable of P, so each S_i would be a process constant $S_i \approx a_i.0$. Now, for such S_i to have two prefixes a_{j_1} and a_{j_2}, both indices would have to be identical, $j_1 = i = j_2$, which contradicts our assumption. Thus, $i = n$. Further, being a pearl, each of the prefixes a_{j_1} and a_{j_2} must be guarded in $P = w(\tilde{S})$ behind some occurrence of a shell variable $y \in Z$. Since $i = n$ and $j_1, j_2 \neq n$, this guard cannot be w but must be an occurrence inside $S_i = S_n$. In other words, S_n must contain at least one variable from Z. But this means that $S_n \trianglelefteq P$ is a pearl in Z, as desired.

  Now we can proceed with our treatment of the present inductive case. The identity $\xi_n^{z'}(U) \equiv \xi_n^{Y}(P) \equiv \xi_n^{Y}(w)[\xi_n^{Y}(\tilde{S})]$ comes down to

  $$\xi_n^{z'}(U) \;\equiv\; E_{n-1}^{n}[\xi_n^{Y_1}(S_1), \xi_n^{Y_2}(S_2), \ldots, \xi_n^{Y_n}(S_n)]. \tag{8}$$

  There are two ways in which this can happen:

  - First, U might be of the shape $U \equiv z'(\tilde{V})$, for $\tilde{V} = V_1, V_2, \ldots, V_n$ and $\xi_n^{Y_i}(S_i) \equiv \xi_n^{z'}(V_i)$. We argue in a similar manner as for summation:

    If $z' \ntrianglelefteq V_i$ for any $i \leq n$, then we can put $X_i =_{\rm df} Y_i$ and $Z'_i =_{\rm df} \emptyset$ to get $Y_i = X_i \uplus Z'_i$ and $V_i \equiv \xi_n^{X_i}(S_i\{z'/Z'_i\})$ quite trivially.

    If $z' \trianglelefteq V_i$, then because of the essential nature of all occurrences of z' (in U and thus also in V_i), we can exploit the above auxiliary result and conclude that $i = n$ and that $S_n \trianglelefteq P$ is a pearl in shell variables Z. We can thus invoke the induction hypothesis on the identity $\xi_n^{Y_i}(S_n) \equiv \xi_n^{z'}(V_n)$ to get a partition $Y_n = X_n \uplus Z'_n$ and $V_n \equiv \xi_n^{X_n}(S_n\{z'/Z'_n\})$.

Taking all together, we can construct X_i, Z'_i with $Y_i = X_i \uplus Z'_i$ and $V_i \equiv \xi_n^{X_i}(S_i\{z'/Z'_i\})$, for all $i \leq n$. Now, define $X =_{\rm df} X_1 \uplus X_2 \uplus \cdots \uplus X_n$ and $Z' =_{\rm df} Z'_1 \uplus Z'_2 \uplus \cdots \uplus Z'_n \uplus \{w\}$, for which we get $Y = X \uplus Z'$ as well as

$$\begin{array}{ll} \xi_n^X(P\{z'/Z'\}) & \\ \quad \equiv & \xi_n^X(w(\tilde{S})\{z'/Z'\}) \\ \quad \equiv & \xi_n^X(z'(\tilde{S}\{z'/Z'\})) \\ \quad \equiv & z'(\xi_n^X(S_1)\{z'/Z'\}, \xi_n^X(S_2)\{z'/Z'\}, \ldots, \xi_n^X(S_n)\{z'/Z'\}) \\ \quad \equiv & z'(\xi_n^{X_1}(S_1)\{z'/Z'_1\}, \xi_n^{X_2}(S_2)\{z'/Z'_2\}, \ldots, \xi_n^{X_n}(S_n)\{z'/Z'_n\}) \\ \quad \equiv & z'(V_1, V_2, \ldots, V_n) \;\equiv\; U. \end{array}$$

Here, we used the fact that $z' \notin X \subseteq Y$, because $z' \ntrianglelefteq P$ and thus $z' \notin Y$.

– Second, it may be that $U \equiv \mu x.\, U'$ and that the recursion μx matches the top-level recursion of E^n_{n-1}, i.e., (8) can be refined to

$$\xi_n^{z'}(U') \equiv \xi_n^{Y_1}(S_1) + \tau.\, E^n_{n-2}[\xi_n^{Y_2}(S_2), \ldots, \xi_n^{Y_n}(S_n), x] \tag{9}$$

Since instances of z' in U' cannot generate a summation, τ-prefix, or any of the sub-terms in E^n_{n-2}, we must have

$$U' \equiv V_1 + \tau.\, E^n_{n-2}[V_2, \ldots, V_n, x]\,,$$

for $\tilde{V} = V_1, V_2, \ldots, V_n$ such that $\xi_n^{z'}(V_i) \equiv \xi_n^{Y_i}(S_i)$. In particular, then, $U \equiv E^n_{n-1}[\tilde{V}]$. All occurrences of z' in any V_i must be essential. As above, if $z' \ntrianglelefteq V_i$, we put $X_i =_{\rm df} Y_i$ and $Z'_i =_{\rm df} \emptyset$; otherwise, if $z' \trianglelefteq V_i$, we must have $i = n$ and S_n being a pearl, which can be handled by induction hypothesis. Overall, we get X_i, Z'_i with $Y_i = X_i \uplus Z'_i$ and $V_i \equiv \xi_n^{X_i}(S_i\{z'/Z'_i\})$, for all $i \leq n$. Here, we put variable w into the set $X =_{\rm df} X_1 \uplus X_2 \uplus \cdots \uplus X_n \uplus \{w\}$ and, with $Z' =_{\rm df} Z'_1 \uplus Z'_2 \uplus \cdots \uplus Z'_n$, we obtain $Y = X \uplus Z'$ as well as

$$\begin{array}{ll} \xi_n^X(P\{z'/Z'\}) & \\ \quad \equiv & \xi_n^X(w(\tilde{S})\{z'/Z'\}) \\ \quad \equiv & \xi_n^X(w(\tilde{S}\{z'/Z'\})) \\ \quad \equiv & E^n_{n-1}[\xi_n^X(S_1)\{z'/Z'\}, \xi_n^X(S_2)\{z'/Z'\}, \ldots, \xi_n^X(S_n)\{z'/Z'\}] \\ \quad \equiv & E^n_{n-1}[\xi_n^{X_1}(S_1)\{z'/Z'_1\}, \xi_n^{X_2}(S_2)\{z'/Z'_2\}, \ldots, \xi_n^{X_n}(S_n)\{z'/Z'_n\}] \\ \quad \equiv & E^n_{n-1}[V_1, V_2, \ldots, V_n] \;\equiv\; U. \end{array}$$

- The final case of the induction is where $P \equiv \mu x.\, P'$ and thus $\mu x.\, \xi_n^Y(P') \equiv \xi_n^{z'}(U)$. Again we have two different scenarios to look at.

  The easy one is where $U \equiv \mu x.\, U'$ and $\xi_n^Y(P') \equiv \xi_n^{z'}(U')$. Clearly, all occurrences of z' in U' must be essential and P' be a pearl in shell variables Z. Here the induction hypothesis can be applied in a straightforward way as for action prefixes.

The situation is more subtle if the recursion μx is matched in U by a top level occurrence of the variable z', i.e., if $U \equiv z'(\tilde{V})$ and

$$\xi_n^Y(P') \equiv \xi_n^{z'}(V_1) + \tau.\, E_{n-2}^n[\xi_n^{z'}(V_2), \ldots, \xi_n^{z'}(V_n), x]. \tag{10}$$

In the following we show that this cannot happen. Observe that (10) is a symmetrical situation to (9) with the roles of P and U interchanged, and in the same manner as before we conclude that

$$P' \equiv S_1 + \tau.\, E_{n-2}^n[S_2, \ldots, S_n, x]\,,$$

for $\tilde{S} = S_1, S_2, \ldots, S_n$ such that $\xi_n^Y(S_i) \equiv \xi_n^{z'}(V_i)$. The point is that no instantiation of a shell variable from Y in P' can produce any of the sub-expressions of the right-hand side of (10) other than sub-expressions of the $\xi_n^{z'}(V_i)$. In particular, $P \equiv E_{n-1}^n[\tilde{S}]$. At this point we recall that the occurrence $z'(\tilde{V})$ is not arbitrary but essential, i.e., *rk*-compatible in U with some $z \in Z \setminus Y$. Let $z(\tilde{Q}) \trianglelefteq U$ for $\tilde{Q} = Q_1, Q_2, \ldots, Q_n$ be the (single) occurrence of $z \neq z'$ in U. One can show that compatibility of $\tilde{V}$ with $\tilde{Q}$ implies compatibility of $\xi_n^{z'}(\tilde{V})$ with $\xi_n^{z'}(\tilde{Q})$. This means that there exist contexts $\tilde{C} = C_1, C_2, \ldots, C_n$ of rank *rk* expressions together with $\tilde{E}^V = E_1^V, E_2^V, \ldots, E_{rk}^V$ and $\tilde{E}^Q = E_1^Q, E_2^Q, \ldots, E_{rk}^Q$ such that, for all $1 \leq i \leq n$,

$$C_i[\tilde{E}^Q] \;\equiv\; \xi_n^{z'}(Q_i) \tag{11}$$

$$C_i[\tilde{E}^V] \;\equiv\; \xi_n^{z'}(V_i). \tag{12}$$

These compatibility identities (11) and (12) constitute a serious constraint on $\xi_n^{z'}(\tilde{V})$, since the expressions $\xi_n^{z'}(\tilde{Q})$ are actually taken from pearl P. Specifically, $z(\tilde{Q}) \trianglelefteq U$ implies $z(\xi_n^{z'}(\tilde{Q})) \equiv \xi_n^{z'}(z(\tilde{Q})) \trianglelefteq \xi_n^{z'}(U) \equiv \xi_n^Y(P)$. This means that there must be an occurrence $z(\tilde{R}) \trianglelefteq P$, for $\tilde{R} = R_1, R_2, \ldots, R_n$ and $\xi_n^{z'}(Q_i) \equiv \xi_n^Y(R_i)$. Since P is a pearl in Z and w a shell variable, we have $R_i \approx a_i.0$ for all $1 \leq i \leq n-1$. In particular, no R_i contains any of the variables from Z, whence $\xi_n^Y(R_i) \equiv R_i$ and thus $\xi_n^{z'}(Q_i) \approx a_i.0$.

Now, here is the problem: because of (11), each of the $n-1$ different process constants $a_i.0 \approx C_i[\tilde{E}^Q]$ ($i \leq n-1$) is produced from its context C_i using "only" *rk* $< n-1$ fixed call-back parameters $\tilde{E}^Q$. These *rk* call-backs $E_1^Q, E_2^Q, \ldots, E_{rk}^Q$ are not sufficient to generate all the $n-1$ different behaviours $a_i.0$, unless at least one of the contexts C_j already contains one the actions a_j, where $1 \leq j \leq n-1$. But then the associated identity $C_j[\tilde{E}^V] \equiv \xi_n^{z'}(V_j)$ from (12) can only hold, if V_j contains this action prefix a_j and thus also S_j because of $\xi_n^Y(S_j) \equiv \xi_n^{z'}(V_j)$.

Let us take a closer look at the action prefix a_j in S_j and C_j. The action prefix a_j in S_j must be j-guarded in pearl $P \equiv E_{n-1}^n[\tilde{S}]$ by some shell variable $w \in Z$. Because of the way sub-expression S_j "sits" inside P, this implies that S_j itself must contain this guarding shell variable. Therefore, sub-expression S_j is a pearl. This follows from the way in which S_j is a subexpression of P as well as our observations about the structure of pearls made in Sec. A.3.

To sum up, we have $C_j[\tilde{E}^V] \equiv \xi_n^Y(S_j)$, where C_j contains a_j and S_j is a pearl in shell variables Z. Also, $C_j[\tilde{E}^Q] \approx a_j.0$ as we have shown above, so it is clear that C_j cannot include any of the other actions a_i $(i \neq j)$ of pearl S_j. Moreover, C_j cannot mention any shell variable from Z. Thus, all shell variables in S_j are in Y. Suppose $w(T_1, T_2, \ldots, T_n) \trianglelefteq S_j$ is the j-guard of a_j in S_j with shell variable $w \in Y$. Then, $E^n_{n-1}[\xi_n^Y(T_1), \xi_n^Y(T_2), \ldots, \xi_n^Y(T_n)] \equiv \xi_n^Y(w(T_1, T_2, \ldots, T_n)) \trianglelefteq \xi_n^Y(S_j) \equiv C_j[\tilde{E}^V]$. By assumption, the a_j we have picked occurs in $\xi_n^Y(T_j)$ on the one hand and at the same time in C_j on the other hand. Since S_j is a pearl, $\xi_n^Y(T_i) \approx a_i.0$ for all $1 \leq i \leq n-1$. Now C_j does not contain any of the actions a_j for $j \neq i$, which means that these other prefixes must be generated in $C_j[\tilde{E}^V]$ through the call-backs $\tilde{E}^V$. Because of the special structure of E^n_{n-1} (cf. Lemma 4.1), this requires at least $n-2$ call-backs $E_1^V, E_2^V, \ldots, E_{rk}^V$, viz. one to cover each of the a_i in $\xi_n^Y(T_i)$ for $i \neq j$. But by assumption, there are only *rk* $< n-2$ call-backs available, which is a contradiction. This shows that identity (10) cannot hold and our final recursive case is completed.

∎

Proposition 4.4. *Let E be a scheme of maximal recursion depth rd in which all free variables have maximal rank rk. Suppose rd < 2 and rk $< n-2$, or rd $< n-1$ and rk < 2. Then, every instantiation θ such that $\theta(E)$ is an n-noose can be factorised as $\theta = \xi_n^Z \circ \theta'$ for some instantiation θ' and rank n shell variables Z in such as way that $\theta'(E)$ is a shell in shell variables Z.*

Proof: By assumption, there exists an n-shell S with (nonempty set of) shell variables Z such that $\xi_n^Z(S) \equiv \theta(E)$. The n-shell S must contain a 0-closed pearl, say P, i.e., $S \equiv S'[P]$ for some rank 1 expression S'. We may assume without loss of generality that each shell variable $z \in Z$ has a singleton loop occurrence in P and that \$1 is singleton in S', i.e., \$1 $\trianglelefteq_1 S'$. Pick any shell variable $z_1 \in Z$, i.e., $Z =_{\mathrm{df}} Z_1 \uplus \{z_1\}$. Defining $u =_{\mathrm{df}} \xi_n^Z(S')$ and

$$U_1 =_{\mathrm{df}} u[\xi_n^{Z_1}(P)] \equiv \xi_n^Z(S')[\xi_n^{Z_1}(P)]$$

we find

$$\theta(E) \equiv \xi_n^Z(S'[P]) \equiv \xi_n^{z_1}(\xi_n^Z(S')[\xi_n^{Z_1}(P)]) \equiv \xi_n^{z_1}(U_1)\,,$$

where z_1 has a singleton loop occurrence in U_1. Now apply Lemma A.18 which yields a scheme U_1' and instantiation θ_1' such that

$$\begin{aligned} \theta &= \xi_n^{z_1} \circ \theta_1' \\ \theta_1'(E) &\equiv U_1'\{z_1/z_1'\} \\ U_1 &\equiv U_1'\{E^n_{n-1}/z_1'\}\,. \end{aligned}$$

In particular, this means that $u[\xi_n^{Z_1}(P)] \equiv U_1 \equiv U_1'\{E^n_{n-1}/z_1'\} \equiv \xi_n^{z_1'}(U_1')$. Also, all occurrences of z in $\theta'(E)$ are *rk*-compatible in $\theta_1'(E) \equiv U_1'\{z_1/z_1'\}$, and thus z_1, z_1' are *rk*-compatible in U_1'.

Since \$1 $\trianglelefteq_1 u$ and $\xi_n^{Z_1}(P)$ is free for \$1 in u (it is actually 0-closed, hence this is trivial), the Decomposition Lemma (Lem. A.17) implies the existence of a scheme V_1 and rank 1 expression U_1'' with \$1 $\trianglelefteq U_1''$, $U_1' \equiv U_1''[V_1]$, $u \equiv \xi_n^{z_1'}(U_1'') \equiv U_1''\{E_{n-1}^n/z_1'\}$ and

$$\xi_n^{z_1'}(V_1) \equiv \xi_n^{Z_1}(P)\,. \tag{13}$$

Since z, z' are *rk*-compatible in U_1' and $U_1' \equiv U_1''[V_1]$, all the occurrences of z_1' and z_1 in V_1 are rk-compatible. Now, rk $< n-2$, so we can apply the Pearl Abstraction Lemma (Lem. A.19) to "divide" both sides of the identity (13) by $\xi_n^{z_1'}$. More precisely, there is a partitioning $Z_1 = X_1 \uplus Z_1'$ such that $V_1 \equiv \xi_n^{X_1}(P\{z_1'/Z_1'\})$. This gives

$$\begin{array}{rcl} \theta_1'(E) & \equiv & U_1'\{z_1/z_1'\} \\ & \equiv & U_1''[\xi_n^{X_1}(P\{z_1'/Z_1'\})]\{z_1/z_1'\} \\ & \equiv & U_1''\{z_1/z_1'\}[\xi_n^{X_1}(P\{z_1/Z_1'\})]. \end{array}$$

If $X_1 = \emptyset$, then we get $\theta_1'(E) \equiv U_1''\{z_1/z_1'\}[P\{z_1/Z_1'\}]$ which means we are done since, on the one hand, $\theta = \xi_n^{z_1} \circ \theta_1'$ and, on the other hand, $P\{z_1/Z_1'\} \trianglelefteq \theta_1'(E)$, where $P\{z_1/Z_1'\}$ is a pearl. Thus, $\theta_1'(E)$ is an n-shell.

If $X_1 \neq \emptyset$, say $X_1 = Z_2 \uplus \{z_2\}$, then we continue. We may use \$1 $\trianglelefteq U_1''$ but $z_2 \not\trianglelefteq U_1''$. The latter holds because no variable from Z_1 appears in U_1, whence not in U_1' and thus not in U_1''. From this we then obtain

$$\begin{array}{rcl} \theta_1'(E) & \equiv & U_1''\{z_1/z_1'\}[\xi_n^{X_1}(P\{z_1/Z_1'\})] \\ & \equiv & \xi_n^{z_2}(U_1''\{z_1/z_1'\}[\xi_n^{Z_2}(P_1)]) \\ & \equiv & \xi_n^{z_2}(U_2), \end{array}$$

where $U_2 =_{\mathrm{df}} U_1''\{z_1/z_1'\}[\xi_n^{Z_2}(P_1)]$ and $P_1 =_{\mathrm{df}} P\{z_1/Z_1'\}$. With these abbreviations we can continue the factorisation inductively with Z_2 in place of Z_1, U_2 instead of U_1 and θ_1' for θ. Note that z_2 has a singleton loop occurrence in $\xi_n^{Z_2}(P)$ and \$1 $\trianglelefteq_1 U_1''$, whence z_2 has a singleton loop occurrence in U_2. We can therefore continue to apply the Factorisation Lemma (Lem. A.18) in this way to get another substitution θ_2' and scheme U_2' such that

$$\begin{array}{rcl} \theta_1' & = & \xi_n^{z_2} \circ \theta_2' \\ \theta_2'(E) & \equiv & U_2'\{z_2/z_2'\} \\ U_2 & \equiv & U_2'\{E_{n-1}^n/z_2'\}. \end{array}$$

The last identity means

$$U_1''\{z_1/z_1'\}[\xi_n^{Z_2}(P_1)] \equiv U_2 \equiv U_2'\{E_{n-1}^n/z_2'\}.$$

Again, the Decomposition Lemma (Lem. A.17) and Abstraction Lemma (Lem. A.19) imply the existence of expressions V_2, U_2'' (free variables z_1, z_1', z_2, z_2') such that $U_2' \equiv$

$U_2''[V_2]$ and $U_1''\{z/z'\} \equiv \xi_n^{z_2'}(U_2'') \equiv U_2''\{E_{n-1}^n/z_2'\}$, together with a partitioning $Z_2 = X_2 \uplus Z_2'$ such that $V_2 \equiv \xi_n^{X_2}(P_1\{z_2'/Z_2'\})$. From this we obtain

$$\begin{array}{rcl} \theta_2'(E) & \equiv & U_2'\{z_2/z_2'\} \\ & \equiv & U_2''[\xi_n^{X_2}(P_1\{z_2'/Z_2'\})]\{z_2/z_2'\} \\ & \equiv & U_2''\{z_2/z_2'\}[\xi_n^{X_2}(P_1\{z_2/Z_2'\})] \\ & \equiv & \xi_n^{z_3}(U_2''\{z_2/z_2'\}[\xi_n^{Z_3}(P_2)]) \\ & \equiv & \xi_n^{z_3}(U_3), \end{array}$$

where $U_3 =_{\mathrm{df}} U_2''\{z_2/z_2'\}[\xi_n^{Z_3}(P_2)]$ and $P_2 =_{\mathrm{df}} P_1\{z_2/Z_2'\}$, assuming $X_2 \neq \emptyset$ and $X_2 = Z_3 \uplus \{z_3\}$. Continuing in this way by induction on the remaining shell variables Z_i, we construct instantiations $\theta_1', \theta_2', \theta_3', \ldots, \theta_k'$ and schemes

$$U_1, U_1', U_1'', U_2, U_2', U_2''', \ldots, U_k, U_k', U_k''$$

such that (for $i = 1, \ldots, k-1$):

$$\begin{array}{rcl} \theta_i' & = & \xi_n^{z_{i+1}} \circ \theta_{i+1}' \\ \theta_{i+1}'(E) & \equiv & U_{i+1}'\{z_{i+1}/z_{i+1}'\} \\ U_{i+1}' & \equiv & U_{i+1}''[V_{i+1}] \;\equiv\; U_{i+1}''[\xi_n^{X_{i+1}}(P_i\{z_{i+1}/z_{i+1}'\})] \\ U_{i+1} & \equiv & U_i''\{z_i/z_i'\}[\xi_n^{Z_{i+1}}(P_i)] \end{array}$$

$$P_{i+1} \equiv P_i\{z_{i+1}/z_{i+1}'\} \qquad X_i = Z_{i+1} \uplus \{z_{i+1}\} \qquad Z_i = X_i \uplus Z_i'.$$

Finally, for $i = k$ we reach the situation where $X_k = \emptyset$, $Z_i = Z_i'$ and

$$\begin{array}{rcl} \theta_k'(E) & \equiv & U_k'\{z_k/z_k'\} \\ & \equiv & U_k''[\xi_n^{X_k}(P_{k-1})\{z_k'/Z_k'\}]\{z_k/z_k'\} \\ & \equiv & U_k''\{z_k/z_k'\}[\xi_n^{X_k}(P_{k-1}\{z_k/Z_k'\})] \\ & \equiv & U_k''\{z_k/z_k'\}[\xi_n^{X_k}(P\{z_i/Z_i'\}_{i=1}^k)] \\ & \equiv & U_k''\{z_k/z_k'\}[P\{z_i/Z_i'\}_{i=1}^k)]. \end{array}$$

Since $\$1 \trianglelefteq U_k''$ we find that $\theta_k'(E)$ contains $P' \equiv P\{z_i/Z_i'\}_{i=1}^k$, which is nothing but the 0-closed pearl P with some shell variables identified. Further, since

$$\theta = \xi_n^{z_1} \circ \cdots \circ \xi_n^{z_k} \circ \theta_k' = \xi_n^Z \circ \theta_k'$$

and $\theta(E) \approx \mathbf{A}_n$, this means in fact that $\theta_k'(E)$ is an n-shell with shell variables $Z' = \{z_1, z_2, \ldots, z_k\}$. This completes the proof of Proposition 4.4. ∎

A.7 Proof of Theorem 4.8

Proposition 4.7. *If Bloom/Ésik's rule GA is sound for $\approxeq$, then it also sound for $\approxeq_n$, for all $n \geq 5$.*

Proof: First, recall Bloom and Ésik's rank 2 rule

$$\text{GA}\ \frac{\mu x.\, w_1(x,x) = \mu x.\, w_2(x,x)}{\mu x.\, w_1(x,x) = \mu x.\, w_2(x, \mu y.\, w_1(x,y))}$$

Assuming that GA is sound for $\approxeq$, we show that it is also sound for $\approxeq_n$ with $n \geq 2$. For convenience, let scheme $\mu x.\, w_2(x, \mu y.\, w_1(x,y))$ be abbreviated as E and $\mu x.\, w_j(x,x)$ as E_j in the following. Suppose $\theta(E_1) \approxeq_n \theta(E_2)$, from which we must prove that $\theta(E_1) \approxeq_n \theta(E)$. Since GA is sound for $\approxeq$, this reduces to the statements:

($\Leftarrow$) If $\theta(E)$ is an n-noose, then *one of* $\theta(E_1)$ *or* $\theta(E_2)$ is an n-noose, and

($\Rightarrow$) If *both* $\theta(E_1)$ *and* $\theta(E_2)$ are n-nooses, then $\theta(E)$ is an n-noose,

under the assumption that $\theta(E) \approxeq \theta(E_1) \approxeq \theta(E_2) \approxeq \mathbf{A}_n$. Without loss of generality, it suffices to prove these statements for non-trivial instantiations, specifically those θ such that $\$2 \trianglelefteq \theta(w_2)$. The reason is that in any proof which instantiates GA so that $\theta(w_2)$ does not use call-back \$2, the application of GA is redundant. Both the premise and the conclusion are identical for such θ. In particular, $\theta(E) \equiv \theta(E_2)$ so that ($\Leftarrow$) and ($\Rightarrow$) are trivial. Thus, assume $\$2 \trianglelefteq \theta(w_2)$ henceforth.

($\Leftarrow$) Suppose that $\theta(E)$ is an n-noose with $n \geq 2$, i.e., there is a 0-closed pearl P in shell variable z such that

$$\xi^z_n(P) \quad \trianglelefteq \quad \mu x.\, \theta(w_2)[x, \mu y.\, \theta(w_1)[x,y]]. \tag{14}$$

We distinguish two main cases, depending on whether $\xi^z_n(P)$ is a proper sub-expression of the right-hand side or not:

- Suppose $\xi^z_n(P) \equiv \mu x.\, \theta(w_2)[x, \mu y.\, \theta(w_1)[x,y]]$. Observe that the 0-closed pearl P cannot be of the form $z(\tilde{Q})$ which means that the recursion μx cannot belong to an occurrence of E^n_{n-1} arising from some top-level z in P. Instead, it must be part of P proper, i.e., $P \equiv \mu x.\, Q$ and $\xi^z_n(Q) \equiv \theta(w_2)[x, \mu y.\, \theta(w_1)[x,y]]$, or more compactly,

$$\xi^z_n(Q) \quad \equiv \quad W[\mu y.\, \theta(w_1)[x,y]], \tag{15}$$

where $W =_{\mathrm{df}} \theta(w_2)[x, \$1]$. Note that, by assumption, $\$2 \trianglelefteq \theta(w_2)$ and thus $\$1 \trianglelefteq W$. Let $m \geq 1$ be the number of occurrences of \$1 in W. We apply Lem. A.17 to "divide" both sides of (15) by ξ^z_n. This yields a sequence of expressions $\tilde{V} = V_1, V_2, \ldots, V_m$ and a rank m expression S such that

$$Q \quad \equiv \quad S[\tilde{V}] \tag{16}$$

$$\theta(w_2)[x, \$1] \quad \equiv \quad W \ \equiv \ \xi^z_n(S)[\$1^m] \tag{17}$$

$$\mu y.\, \theta(w_1)[x,y] \quad \equiv \quad \xi^z_n(V_i) \quad (i \leq m). \tag{18}$$

The fact that $P \equiv \mu x. Q$ is a 0-closed pearl in shell variable z means that Q can have at most x and z as (free) variables and the $a_i \in \mathbf{A}_n$ as observable actions. Because of (16), the same must be true of S, which may or may not have call-backs $\$1, \$2, \ldots, \$m$. This means that $\mu x. S[\tilde{x}]$, where sequence $\tilde{x} = x, x, \ldots, x$ is of length m, is not only 0-closed but also pure and thus satisfies property (P1) of a pearl. Note that $\xi^z_n(\mu x. S[\tilde{x}]) \equiv \mu x. \xi^z_n(S)[\tilde{x}] \equiv \mu x. \theta(w_2)[x, x] \equiv \theta(E_2)$ because of (17). So, if we can show that $\mu x. S[\tilde{x}]$ is indeed a 0-closed pearl (in shell variable z), then it follows (by Lem. A.6) that $\theta(E_2)$ is an n-noose.

To this end we prove properties (P2) and (P3) for $\mu x. S[\tilde{x}]$. Consider any occurrence $z(\tilde{T}, U) \trianglelefteq S[\tilde{x}]$ and $\tilde{T} = T_1, T_2, \ldots, T_{n-1}$. Since certainly $z(\tilde{T}, U) \not\trianglelefteq x$, we may factor out (by Lem. A.16) the call-backs of S in $z(\tilde{T}, U) \trianglelefteq S[\tilde{x}]$ to obtain rank m expressions $\tilde{T}' = T'_1, T'_2, \ldots, T'_{n-1}$ and U' such that $T'_i[\tilde{x}] \equiv T_i$, $U'[\tilde{x}] \equiv U$ and $z(\tilde{T}', U') \trianglelefteq S$. Then,

$$z(\tilde{T}'[\tilde{V}], U'[\tilde{V}]) \;\equiv\; z(\tilde{T}', U')[\tilde{V}] \;\trianglelefteq\; S[\tilde{V}] \;\equiv\; Q \;\trianglelefteq\; P.$$

Now we exploit the fact that P is a 0-closed pearl. To start with, by property (P2), each $T'_i[\tilde{V}]$ must be a closed process and $T'_i[\tilde{V}] \approx a_i.0$. We claim that this is only possible if T'_i does not contain any of the call-backs $\$k$ for $k = 1, \ldots, m$. Suppose, by way of contradiction, that $\$k \trianglelefteq T'_i$ in which case $V_k \trianglelefteq T'_i[\tilde{V}]$. Since $T'_i[\tilde{V}] \approx a_i.0$, shell variable z cannot be free in $T'_i[\tilde{V}]$ and hence in V_k. But then, using (18), we argue that $\mu x. V_k \equiv \mu x. \xi^z_n(V_k) \equiv \mu x. \mu y. \theta(w_1)[x, y] \approxeq \mu x. \theta(w_1)[x, x] \equiv \theta(E_1) \approx \mathbf{A}_n$. This means that V_k, as a sub-expression of $T'_i[\tilde{V}]$, would exhibit all actions a_j for $j = 1, \ldots, n-1$, which is impossible due to $T'_i[\tilde{V}] \approx a_i.0$. Thus, we conclude that T'_i makes no call-back at all and, therefore, $T_i \equiv T'_i[\tilde{x}] \equiv T'_i[\tilde{V}] \approx a_i.0$.

Another consequence of 0-closed pearl P is that $U'[\tilde{V}]$ must have a free process variable, either in U' or some call-back V_k. But then, as is easy to see, $U \equiv U'[\tilde{x}]$ must have a free process variable, too. This shows that $\mu x. S[\tilde{x}]$ satisfies property (P2) of a pearl.

Now take property (P3). Since $\xi^z_n(\mu x. S[\tilde{x}]) \equiv \theta(E_2) \approx \mathbf{A}_n$ as observed above, the process constant $\xi^z_n(\mu x. S[\tilde{x}])$ must be able to perform at least one a_i action (for all $i = 1, \ldots, n-1$). Thus, S must contain at least one a_i prefix because no such prefix could be introduced by the instantiation ξ^z_n. But if S has an a_1 prefix, Q must contain one, too, given $Q \equiv S[\tilde{V}]$ from (16). Since $P \equiv \mu x. Q$ is a 0-closed pearl, this prefix a_1 in Q must be 1-guarded by at least one occurrence of z. It is easy to see that the same occurrence of z must 1-guard a_1 in S. This is because the first-order substitution turning S into $S[\tilde{V}]$ cannot make any a_1 in S that is not 1-guarded by any z, miraculously 1-guarded in $S[\tilde{V}]$. Hence, we find that S must contain at least one occurrence of z. The same reasoning can be repeated for any action $a_i \in \mathbf{A}^+_n$, which gives property (P3) to show overall that $\mu x. S[\tilde{x}]$ is a 0-closed pearl and $\theta(E_2)$ an n-noose as claimed.

- Let $\xi_n^z(P) \trianglelefteq \theta(w_2)[x, \mu y.\, \theta(w_1)[x, y]]$. By way of Lem. A.16 we distinguish two further cases depending on whether $\xi_n^z(P)$ starts in $\theta(w_2)$ or in (some occurrence of) the call-back term $\mu y.\, \theta(w_1)[x, y]$.

  Suppose the latter, i.e., $\xi_n^z(P) \trianglelefteq \mu y.\, \theta(w_1)[x, y]$. Again, there are two cases: if $\xi_n^z(P) \equiv \mu y.\, \theta(w_1)[x, y]$ then $\theta(w_1)$ cannot use call-back \$1, for then x would be free in $\xi_n^z(P)$ which is not possible for 0-closed pearl P. But if $\theta(w_1)$ does not mention \$1, we get $\mu y.\, \theta(w_1)[y, y] \equiv \mu y.\, \theta(w_1)[x, y] \equiv \xi_n^z(P)$ from which it follows without difficulties that $\theta(E_1) \equiv \mu x.\, \theta(w_1)[x, x]$ must be n-noose, as n-nooses are preserved under α-conversion.

  It remains for us to consider the case that $\xi_n^z(P) \trianglelefteq \theta(w_1)[x, y]$. Again, since x and y cannot be free in P, this means that $\xi_n^z(P) \trianglelefteq \theta(w_1)$. But then $\xi_n^z(P) \trianglelefteq \mu x.\, \theta(w_1)[x, x] \equiv \theta(E_1)$, which implies, by Lem. A.6, that $\theta(E_1)$ is an n-noose.

- If $\xi_n^z(P) \ntrianglelefteq \mu y.\, \theta(w_1)[x, y]$, then $\xi_n^z(P) \trianglelefteq \theta(w_2)[x, \mu y.\, \theta(w_1)[x, y]]$ implies that there exists a rank 2 expression Q such that $\xi_n^z(P) \equiv Q[x, \mu y.\, \theta(w_1)[x, y]]$ and $Q \trianglelefteq \theta(w_2)$. Since P is 0-closed, this can only hold true if \$1 $\ntrianglelefteq Q$ and also \$2 $\ntrianglelefteq Q$ or \$1 $\ntrianglelefteq \theta(w_1)$. If Q does not perform any call-back, we find $\xi_n^z(P) \equiv Q[x, \mu y.\, \theta(w_1)[x, y]] \equiv Q[x, x] \trianglelefteq \theta(w_2)[x, x] \trianglelefteq \mu x.\, \theta(w_2)[x, x] \equiv \theta(E_2)$. Again, Lem. A.6 implies that $\theta(E_2)$ is an n-noose. In the second case, if \$1 $\ntrianglelefteq Q$ but \$2 $\trianglelefteq Q$ and \$1 $\ntrianglelefteq \theta(w_1)$, we must argue differently. First note that the latter means $\mu y.\, \theta(w_1)[x, y] \equiv \mu y.\, \theta(w_1)[y, y]$. Now define rank 1 expression $W =_{\mathrm{df}} Q[x, \$1]$ for which we get

  $$\xi_n^z(P) \;\; \equiv \;\; W[\mu y.\, \theta(w_1)[y, y]]. \tag{19}$$

  Of course, \$1 $\trianglelefteq W$ since \$2 $\trianglelefteq Q$. Let \$1 $\trianglelefteq_m W$. We apply Lem. A.17 to (19) and obtain m expressions $\tilde{V} = V_1, V_2, \ldots, V_m$ and rank m expression S such that

  $$P \;\; \equiv \;\; S[\tilde{V}] \tag{20}$$
  $$Q[x, \$1] \;\; \equiv \;\; W \; \equiv \; \xi_n^z(S)[\$1^m] \tag{21}$$
  $$\mu y.\, \theta(w_1)[y, y] \;\; \equiv \;\; \xi_n^z(V_i) \quad (i \leq m). \tag{22}$$

  We have to distinguish two cases depending on whether S has a call-back or not. First, if \$k $\ntrianglelefteq S$ for all $k = 1, \ldots, m$ we are quickly done since then $S \equiv S[\tilde{V}] \equiv P$ and further $\xi_n^z(P) \equiv \xi_n^z(S) \equiv \xi_n^z(S)[\tilde{x}] \equiv W[x] \equiv Q[x, x] \trianglelefteq \theta(w_2)[x, x] \trianglelefteq \mu x.\, \theta(w_2)[x, x]$. This proves that $\theta(E_2) \equiv \mu x.\, \theta(w_2)[x, x]$ is an n-noose.

  If \$k $\trianglelefteq S$ for some $k = 1, \ldots, m$, then $V_k \trianglelefteq P$ by (20). We claim that in this case, V_k is a 0-closed pearl. Certainly $z \trianglelefteq V_k$, for otherwise if $z \ntrianglelefteq V_k$, then (22) would give $\mathbf{A}_n \approx \mu y.\, \theta(w_1)[y, y] \equiv \xi_n^z(V_k) \equiv V_k$ which means that V_k can perform any action a_i, for $i = 1, \ldots, n-1$. Because of $V_k \trianglelefteq P$, each action prefix a_i in V_k has an occurrence inside P and thus must be i-guarded by some occurrence of z. However, if V_k itself does not contain the shell variable z, then all the a_i in V_k must sit i-guarded by the same occurrence of z in P. But this is impossible by the construction of pearls. V_k cannot be i-guarded inside P, for $1 \leq i \leq n-1$.

Thus, V_k contains at least one occurrence of shell variable z, which means that V_k is a pearl. Also, observe that, because of (22) and freeness of θ for P, the expression V_k must be 0-closed. Hence, $V_k \trianglelefteq P$ must be pure so that properties (P1) and (P3) hold trivially for V_k. Finally, it is easy to see that every sub-expression of a pearl satisfies (P2). We thus find V_k is a 0-closed pearl with shell variable z and, therefore, $\mu y.\, \theta(w_1)[y,y] \equiv \xi_n^z(V_k)$ is an n-noose. But this means $\theta(E_1) \equiv \mu x.\, \theta(w_1)[x,x]$ is also an n-noose.

(⇒) Let both $\theta(E_j)$ be n-nooses (for $n > 4$). Observe that both expressions E_j with recursion depth 1 and rank 2 can at most peel off the outermost recursion from the noose $\theta(E_j)$. By Prop. 4.4, there are instantiations σ_j such that $\sigma_j(E_j)$ are n-shells and $\theta = \xi_n^z \circ \sigma_j$. We may assume without loss of generality that σ_j has domain $\{w_j\}$ and θ domain $\{w_1, w_2\}$. Hence, both σ_j can be merged into a single substitution σ over domain $\{w_1, w_2\}$ such that $\sigma(E_j) \equiv \sigma_j(E_j)$ and $\theta = \xi_n^z \circ \sigma$. Also, since $\theta(w_2)$ uses call-back \$2 we may assume that σ_i does the same. We claim that $\sigma(E)$ is an n-shell. Since $\xi_n^z(\sigma(E)) \equiv (\xi_n^z \circ \sigma)(E) \equiv \theta(E) \approxeq \mathbf{A}_n$, it suffices to prove that $\sigma(E)$ has a 0-closed pearl as sub-expression. Note that the free variables and observable actions of $\sigma(E) \equiv \mu x.\, \sigma(w_2)[x, \mu y.\, \sigma(w_1)[x,y]]$ are the free variables and observable actions of $\sigma(E_1)$ and $\sigma(E_2)$, i.e., variable z and actions $a_i \in \mathbf{A}_n$.

The property of n-shells tells us that each $\sigma(E_j)$ has a 0-closed pearl $P_j \trianglelefteq \sigma(E_j) \equiv \mu x.\, \sigma(w_j)[x,x]$, which we may assume without loss of generality to be in shell variable z. If one of the pearls satisfies $P_i \trianglelefteq \sigma(w_i)[x,x]$, the argument is simple. Since P_i is 0-closed, in particular $P_i \trianglelefteq \sigma(w_i)$, and thus:

$$\begin{array}{lllllllll} P_1 & \equiv & P_1[x,y] & \trianglelefteq & \sigma(w_1)[x,y] & \trianglelefteq & \mu x.\, \sigma(w_2)[x, \mu y.\, \sigma(w_1)[x,y]] & \equiv & \sigma(E) \\ P_2 & \equiv & P_2[x, \mu y.\, \sigma(w_1)[x,y]] & \trianglelefteq & \sigma(w_2)[x, \mu y.\, \sigma(w_1)[x,y]] & \trianglelefteq & \sigma(E), & & \end{array}$$

where in the first case we exploit that $\sigma(w_2)$ uses call-back \$2. This means that $\sigma(E)$ is an n-shell in either case. Hence, in the sequel we may assume that $P_i \equiv \mu x.\, \sigma(w_i)[x,x]$. Now, if $\sigma(w_1)$ is trivial in the sense that it does not contain call-back \$1, then $P_1' \equiv \mu y.\, \sigma(w_1)[y,y] \equiv \mu y.\, \sigma(w_1)[x,y] \trianglelefteq \sigma(E)$, which means that $\sigma(E)$ is an n-shell (by Lem. A.6). So, we may assume without loss of generality that $\sigma(w_1)[x,y]$ has x free, i.e., \$1 $\trianglelefteq \sigma(w_1)$.

We claim that $\sigma(E)$ is a 0-closed pearl. Clearly, $\sigma(E)$ is pure (condition (P1) of pearls) since both P_i are. Furthermore, it must contain at least two occurrences of z, because it contains $\sigma(w_1)$ and $\sigma(w_2)$ and $z \trianglelefteq \sigma(w_i)$ since $P_i \equiv \sigma(E_i) \equiv \mu x.\, \sigma(w_i)[x,x]$ are pearls by assumption. Also, any occurrence of a_i in $\sigma(E)$ corresponds to an occurrence in one of $P_i \equiv \sigma(E_i)$, which must be i-guarded in P_i. It is obvious that the associated occurrence of i-guard z in P_i is also contained in $\sigma(E)$ and an i-guard of a_i. Hence, $\sigma(E)$ fulfils condition (P3) of pearls.

All that remains to be shown is property (P2), i.e., that in no occurrence $z(\tilde{T}, U) \trianglelefteq \sigma(E)$ for $\tilde{T} = T_1, T_2, \ldots, T_k$, the expression U is 0-closed and the T_i anything other than $T_i \approx a_i.0$. Let such

$$z(\tilde{T}, U) \trianglelefteq \sigma(E) \equiv \mu x.\, \sigma(w_2)[x, \mu y.\, \sigma(w_1)[x, y]]$$

be given. More specifically, this means $z(\tilde{T}, U) \trianglelefteq \sigma(w_2)[x, \mu y.\, \sigma(w_1)[x, y]]$. We consider all possible cases how this can be located (cf. Lemma A.16).

First, suppose $z(\tilde{T}, U) \trianglelefteq \mu y.\, \sigma(w_1)[x, y]$, i.e., $z(\tilde{T}, U) \trianglelefteq \sigma(w_1)[x, y]$. This means that there are schemes $\tilde{T}' = T'_1, T'_2, \ldots, T'_{n-1}$ and U' such that $z(\tilde{T}', U') \trianglelefteq \sigma(w)$, $\tilde{T}'[x, y] \equiv \tilde{T}$ and $U'[x, y] \equiv U$. Then,

$$\begin{array}{rcl} z(\tilde{T}'[x,x], U'[x,x]) & \equiv & z(\tilde{T}, U)\{x/y\} \\ & \trianglelefteq & (\sigma(w_1)[x,y])\{x/y\} \\ & \equiv & \sigma(w_1)[x,x] \\ & \trianglelefteq & \mu x.\, \sigma(w_1)[x,x] \;\equiv\; P_1\,, \end{array}$$

considering that σ is closed and thus $\sigma(w_1)$ does not have y free. Since P_1 is a 0-closed pearl, $U'[x, x]$ cannot be 0-closed, whence also $U \equiv U'[x, y]$ must have a free process variable. Moreover, $T'_i[x, x] \approx a_i.0$ for all $i = 1, \ldots, k$. But this can only be true if indeed $T_i \equiv T'_i[x, y] \approx a_i.0$. This proves condition (P2) in the case that $z(\tilde{T}, U) \trianglelefteq \sigma(w_1)[x, y]$.

The other possibility is that there exists a rank 2 expression $W \trianglelefteq \sigma(w_2)$ with $z(\tilde{T}, U) \equiv W[x, \mu y.\, \sigma(w_1)[x, y]]$. Then, obviously, $W \equiv z(\tilde{W}, V)$ for some $\tilde{W} = W_1, W_2, \ldots, W_{n-1}$ and V with

$$T_i \equiv W_i[x, \mu y.\, \sigma(w_1)[x, y]] \text{ and } U \equiv V[x, \mu y.\; \sigma(w_1)[x, y]].$$

Now we take into account that

$$\begin{array}{rcl} z(\tilde{W}[x,x], V[x,x]) & \equiv & z(\tilde{W}, V)[x,x] \\ & \equiv & W[x,x] \\ & \trianglelefteq & \sigma(w_2)[x,x] \\ & \trianglelefteq & \mu x.\, \sigma(w_2)[x,x] \;\equiv\; P_2. \end{array}$$

Since P_2 is a 0-closed pearl, $V[x, x]$ must have a free process variable. This may be because V contains call-back \$1 or \$2, thereby making x free in $V[x, x]$, or because some other variable y is free in V. In any case, $U \equiv V[x, \mu y.\, \sigma(w_1)[x, y]]$ must have a free process variable, when considering that $\sigma(w_1)[x, y]$ has x free since \$1 $\trianglelefteq \sigma(w_1)$. The pearl property of P_2 also implies that $W_i[x, x] \approx a_i.0$, which immediately lets us conclude that no W_i can have a call-back \$1 or \$2. But then $T_i \equiv W_i[x, \mu y.\, \sigma(w_1)[x, y]] \equiv W_i[x, x] \equiv a_i.0$. This finally proves condition (P2) for $\sigma(E)$ in case $z(\tilde{T}, U)$ is a sub-term of $\sigma(w_2)[x, \mu y.\, \sigma(w_1)[x, y]]$ but not of $\mu y.\, \sigma(w_1)[x, y]$. Thus, $\sigma(E)$ is a 0-closed pearl as claimed. ∎

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